Lectures on

THE COINTEGRATED VECTOR AUTOREGRESSIVE MODEL

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LECTURE 3

Asymptotic Analysis of the I(1) Model and various extensions

1. Asymptotic properties of the process x_t
2. Test for rank
3. Finite sample modifications
4. Asymptotic distribution of $\hat{\beta}$
5. Conclusion
6. Rational Expectations and the VAR
7. Seasonal cointegration
8. Models for Explosive Roots
9. The I(2) model
10. Cofractional error correction model
11. Controlling Inflation
12. Conclusion

1. Asymptotic properties of the process x_t

Consider the model $\Delta x_t = \varepsilon_t, \varepsilon_t$ i.i.d. $(0, \Omega)$ or $x_t = x_0 + \sum_{i=1}^t \varepsilon_i$. Definition of multivariate Brownian motion $W(u),\ u \in [0,1]$ and convergence of moments

$$\begin{aligned}
T^{-1/2} x_{[Tu]} &= T^{-1/2} \sum_{i=1}^{[Tu]} \varepsilon_i \xrightarrow{d} W(u) \\
T^{-1} S_{11} &= T^{-2} \sum_{t=1}^{T} x_{t-1} x_{t-1}' = T^{-1} \sum_{t=1}^{T} \frac{x_{t-1}}{\sqrt{T}} \frac{x_{t-1}'}{\sqrt{T}} \xrightarrow{d} \int_0^1 W(u) W(u)' du \\
S_{10} &= T^{-1} \sum_{t=1}^{T} x_{t-1} \varepsilon_t' = \sum_{t=1}^{T} \frac{x_{t-1}}{\sqrt{T}} \frac{\Delta x_{t-1}'}{\sqrt{T}} \xrightarrow{d} \int_0^1 W(dW)' \\
S_{00} &= T^{-1} \sum_{t=1}^{T} \Delta x_t \Delta x_t' = T^{-1} \sum_{t=1}^{T} \varepsilon_t \varepsilon_t' \xrightarrow{P} \Omega
\end{aligned}$$

2. Test for rank

In the model $\Delta x_t = \Pi x_{t-1} + \varepsilon_t$, the test for $\Pi = 0$ is the Dickey-Fuller test

$$
\begin{aligned}
-2\log LR(\Pi &= 0) = -T\log\frac{\det(S_{00}-S_{01}S_{11}^{-1}S_{10})}{\det(S_{00})} \\
&= -T\log(\det(I_p - S_{00}^{-1}S_{01}S_{11}^{-1}S_{10})) \\
&\approx tr\{S_{00}^{-1}S_{01}(T^{-1}S_{11})^{-1}S_{10}\}
\end{aligned}
$$

which converges ($B = \Omega^{-1/2}W$ is standardized Brownian motion)

$$
\begin{aligned}
&\xrightarrow{d} tr\{\Omega^{-1}\int_0^1 (dW)W'\left(\int_0^1 WW'\right)^{-1}\int_0^1 W(dW)'\} \\
&\xrightarrow{d} tr\{\int_0^1 (dB)B'\left(\int_0^1 BB'\right)^{-1}\int_0^1 B(dB)'\}
\end{aligned}
$$

The Dickey-Fuller distribution (of the trace test) with p degrees of freedom. Tabulated by simulation

TEST FOR RANK (CONTINUED)

Model without deterministic terms

In the model

$$\mathcal{H}_p : \Delta x_t = \Pi x_{t-1} + \sum_{i=1}^{k-1} \Gamma_i \Delta x_{t-i} + \varepsilon_t$$

the test for

$$\mathcal{H}_r : \Pi = \alpha\beta' \text{ of rank } r$$

satisfies

$$-2 \log LR(\mathcal{H}_r|\mathcal{H}_p) = -T \sum_{i=r+1}^{p} \log(1 - \hat{\lambda}_i) \xrightarrow{d} tr\{\int_0^1 (dB)B' \left(\int_0^1 BB'\right)^{-1} \int_0^1 B(dB)'\}$$

where B is standard Brownian motion with $p - r$ degrees of freedom equal to the number of common trends. No parameter dependence in limit distribution.

Modification of limit distribution for model with deterministic terms

In the model

$$\mathcal{H}_p : \Delta x_t = \Pi x_{t-1} + \sum_{i=1}^{k-1} \Gamma_i \Delta x_{t-i} + \mu_0 + \mu_1 t + \varepsilon_t$$

where $\Pi = \alpha\beta'$ and $\mu_1 = \alpha\beta_1'$ (cointegration and no quadratic trend) we find

$$\mathcal{H}_r : \Delta x_t = \alpha \begin{pmatrix} \beta \\ \beta_1 \end{pmatrix}' \begin{pmatrix} x_{t-1} \\ t \end{pmatrix} + \sum_{i=1}^{k-1} \Gamma_i \Delta x_{t-i} + \mu_0 + \varepsilon_t$$

$$-2 \log LR(\mathcal{H}_r | \mathcal{H}_p) = -T \sum_{i=r+1}^{p} \log(1 - \hat{\lambda}_i) \xrightarrow{d} tr\{ \int_0^1 (dB)F' \left(\int_0^1 FF' \right)^{-1} \int_0^1 F(dB)' \}$$

$$F(u) = \begin{pmatrix} B(u) - \bar{B} \\ u - \frac{1}{2} \end{pmatrix}, \quad \bar{B} = \int_0^1 B(u)du$$

Does not depend on the parameters, only on the type of deterministic term.

Limit distribution for rank test in model

$$\mathcal{H}_p : \Delta x_t = \Pi x_{t-1} + \sum_{i=1}^{k-1} \Gamma_i \Delta x_{t-i} + \Phi d_t + \varepsilon_t$$

$$-2 \log LR(\mathcal{H}_r | \mathcal{H}_p) \xrightarrow{d} tr\{\int_0^1 (dB)F' \left(\int_0^1 FF'\right)^{-1} \int_0^1 F(dB)'\}$$

1. No deterministics $(d_t = 0) : F(u) = B(u)$
2. Restricted constant $(\Phi d_t = \alpha\beta_0') : F(u) = \begin{pmatrix} B(u) \\ 1 \end{pmatrix}$
3. Restricted linear term $(\Phi d_t = \mu_0 + \alpha\beta_1' t) : F(u) = \begin{pmatrix} B(u) - \bar{B} \\ u - \frac{1}{2} \end{pmatrix}$

$$x_t = C \sum_{i=1}^{t} (\varepsilon_i + \Phi d_i) + \sum_{i=0}^{\infty} C_i^* (\varepsilon_{t-1} + \Phi d_{t-i}) + A$$

In general the limit distribution will depend on the deterministic terms d_t in the equation unless $T^{-1/2} \sum_{i=1}^{[Tu]} d_i \to 0$.

3. Finite sample modifications

For finite samples the distribution depends on parameters and T.
An approximation to the expected test statistic is

$$\frac{E_{\theta,T}(-2\log LR(\mathcal{H}_r|\mathcal{H}_p))}{E(Trace\,Test)} = a(T,p-r)[1+T^{-1}b(p,\theta)+O(T^{-2})]$$

where $a(T,p-r)$ is simulated and $b(p,\theta)$ is partly analytic and partly simulated.
Bartlett's idea for a better approximation

$$\frac{-2\log LR(\mathcal{H}_r|\mathcal{H}_p)}{E_{\theta,T}(-2\log LR(\mathcal{H}_r|\mathcal{H}_p))} \xrightarrow{d} \frac{Trace\,Test}{E(Trace\,Test)}$$

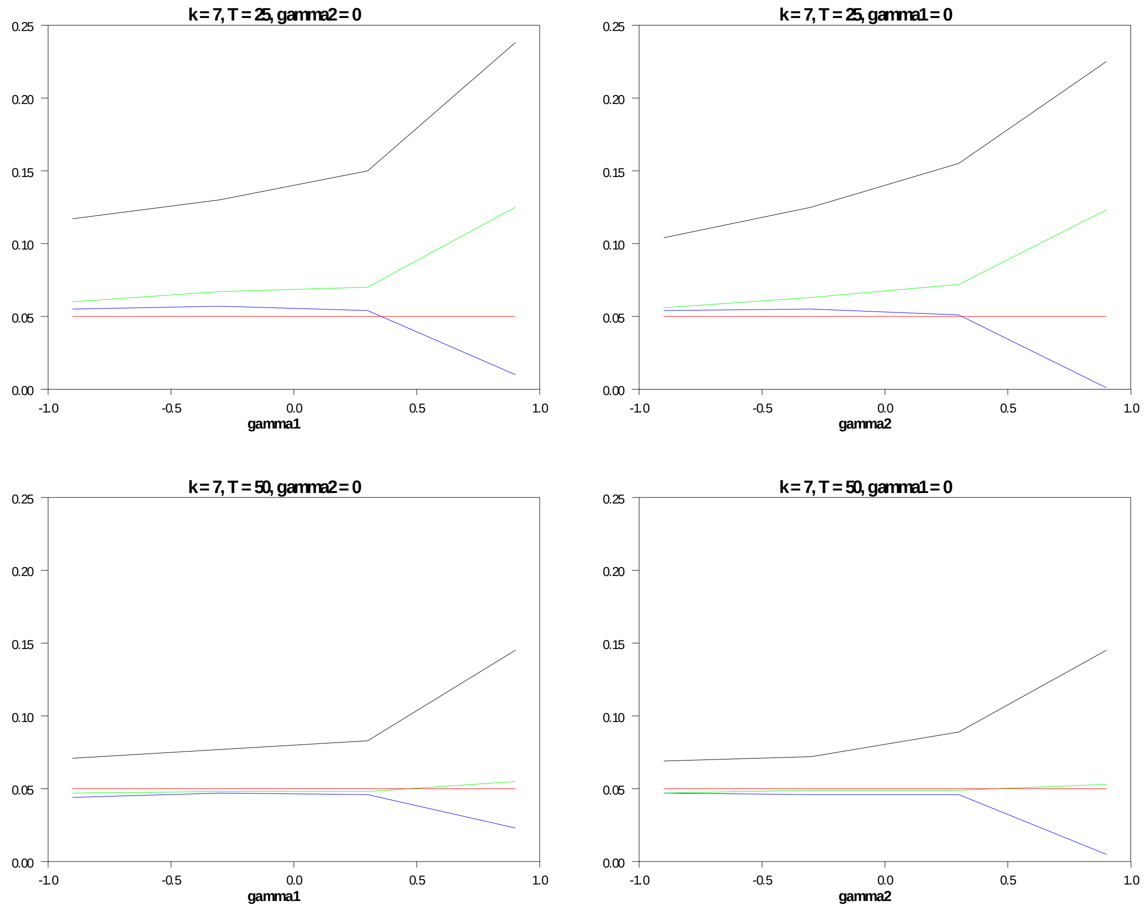
Example

We take $p = 1$ and $k = 2s + 1$ and consider the Dickey-Fuller test

$$\begin{aligned}\Delta x_t &= \pi x_{t-1} + \sum_{i=1}^{2s} \gamma_i \Delta x_{t-i} + \mu_0 + \mu_1 t + \varepsilon_t \\ \mathcal{H}_0 &: \pi = \mu_1 = 0\end{aligned}$$

$$\frac{E_{\theta,T}(-2\log LR(\mathcal{H}_r|\mathcal{H}_p))}{E(Trace\ Test)} \approx [1 + 0.12T^{-1} + 4.05T^{-2}][1 + \frac{1.72}{T}(s + \frac{\sum_{i=1}^{2s} i\gamma_i}{1 - \sum_{i=1}^{2s}\gamma_i})]$$

An extra unit root when $1 - \sum_{i=1}^{2s}\gamma_i = 0$.



5. A small sample correction of the Dickey-Fuller test.
The model is $\Delta x_t = \pi x_{t-1} + \sum_{i=1}^{6} \gamma_i \Delta x_{t-i} + \mu_0 + \mu_1 t + \varepsilon_t$ and the test is $\pi = \mu_1 = 0$. The DGP has $\pi = \mu_1 = 0$ and $\gamma_3 = \cdots = \gamma_6 = 0, \sigma^2 = 1$. The number of simulations is 10,000. Upper panels $T = 25$ and lower panels $T = 50$. Right hand panels $\gamma_2 = 0$, and left hand panels $\gamma_1 = 0$.

Finite sample modifications (cont.)

The cointegrated VAR model with $k = 1$ and $r = 1$

$$\mathcal{H}_p : \Delta x_t = \Pi x_{t-1} + \mu_1 t + \mu_0 + \varepsilon_t,$$

and we test $\Pi = \alpha\beta'$ and $\mu_1 = \alpha\beta_1'$, where α and β are $p \times 1$, that is,

$$\mathcal{H}_r : \Delta x_t = \alpha(\beta' x_{t-1} + \beta_1' t) + \mu_0 + \varepsilon_t.$$

First simulate coefficients $(a_1(p), a_2(p), a_3(p), a_4(p))$ and define the coefficient

$$k(\alpha, \beta, \Omega) = -(1 - \frac{(\beta'\alpha)^2}{\alpha'\Omega^{-1}\alpha\beta'\Omega\beta})\{(2+\beta'\alpha)a_3(p) - 2(4+3\beta'\alpha)\frac{a_4(p)}{(p-1)^2}\} + 2(1+\beta'\alpha)(p-1)\frac{a_4(p)}{(p-1)^2}$$

The correction to the rank test is defined as

$$[1 + T^{-1}a_1(p) + T^{-2}a_2(p)][1 + \frac{1}{T}\frac{1}{\beta'\alpha}k(\alpha, \beta, \Omega)]$$

Problem when $\alpha'\beta$ close to zero, and extra unit root. Either rank zero or $I(2)$.

4. Asymptotic distribution of $\hat{\beta}$

Consider the model with no deterministic terms, $r = 2$, and β identified by

$$\beta = (\underset{p\times 1}{h_1} + \underset{p\times m_1}{H_1}\underset{m_1\times 1}{\varphi_1}, \underset{p\times 1}{h_2} + \underset{p\times m_2}{H_2}\underset{m_2\times 1}{\varphi_2}).$$

THEOREM If ε_t i.i.d. $(0, \Omega)$, then

$$T^{-1/2}x_{[Tu]} \xrightarrow{d} CW = G, \text{ and } T^{-1}S_{11} = T^{-2}\sum_{t=1}^{T} x_{t-1}x_{t-1}' \xrightarrow{d} C\int_0^1 WW'duC' = \mathcal{G}$$

$$T\begin{pmatrix} \hat{\varphi}_1 \\ \hat{\varphi}_2 \end{pmatrix} \xrightarrow{d} \begin{pmatrix} \rho_{11}H_1'\mathcal{G}H_1 & \rho_{12}H_1'\mathcal{G}H_2 \\ \rho_{21}H_2'\mathcal{G}H_1 & \rho_{22}H_2'\mathcal{G}H_2 \end{pmatrix}^{-1} \begin{pmatrix} H_1'\int_0^1 G(dV_1) \\ H_2'\int_0^1 G(dV_2) \end{pmatrix},$$

where $\rho_{ij} = \alpha_j'\Omega^{-1}\alpha_j$, and $V = \alpha'\Omega^{-1}W = (V_1, V_2)'$.
Note that $V = \alpha'\Omega^{-1}W$ is independent of $CW = \beta_\perp(\alpha_\perp'\beta_\perp)^{-1}\alpha_\perp'W$. Hence limit is mixed Gaussian. The estimators of the remaining parameters are asymptotically Gaussian and asymptotically independent of $\hat{\varphi}$ and $\hat{\beta}$.

The observed information $TS_{11} = \sum_{t=1}^{T} x_{t-1}x'_{t-1}$ normalized by T^{-2} converges to a stochastic limit:

$$\mathcal{J}_T = T^{-1}\begin{pmatrix} \rho_{11}H_1'S_{11}H_1 & \rho_{12}H_1'S_{11}H_2 \\ \rho_{21}H_2'S_{11}H_1 & \rho_{22}H_2'S_{11}H_2 \end{pmatrix} \xrightarrow{d} \begin{pmatrix} \rho_{11}H_1'\mathcal{G}H_1 & \rho_{12}H_1'\mathcal{G}H_2 \\ \rho_{21}H_2'\mathcal{G}H_1 & \rho_{22}H_2'\mathcal{G}H_2 \end{pmatrix} = \mathcal{J}.$$

The limit distribution of $\hat{\beta}$ conditional on the limit of the information is Gaussian. Thus

$$\mathcal{J}_T^{1/2}T\begin{pmatrix} \hat{\varphi}_1 \\ \hat{\varphi}_2 \end{pmatrix} \xrightarrow{d} N_{m_1+m_2}(0, I_{m_1+m_2})$$

Thus Wald and likelihood ratio tests on β are asymptotically χ^2.
The scaling is **not** by an estimate of the variance but by an estimate of the **conditional** variance.
It is therefore **not** the asymptotic distribution of $\hat{\beta}$ that is used for inference, but the asymptotic **conditional** distribution given the information.

Example

The simplest example is a cointegrating regression

$$x_{1t} = \theta x_{2t-1} + \varepsilon_{1t}, \text{ and } \Delta x_{2t} = \varepsilon_{2t}$$

where ε_t are i.i.d. Gaussian and $\{\varepsilon_{1t}\}$ and $\{\varepsilon_{2t}\}$ are independent. Then

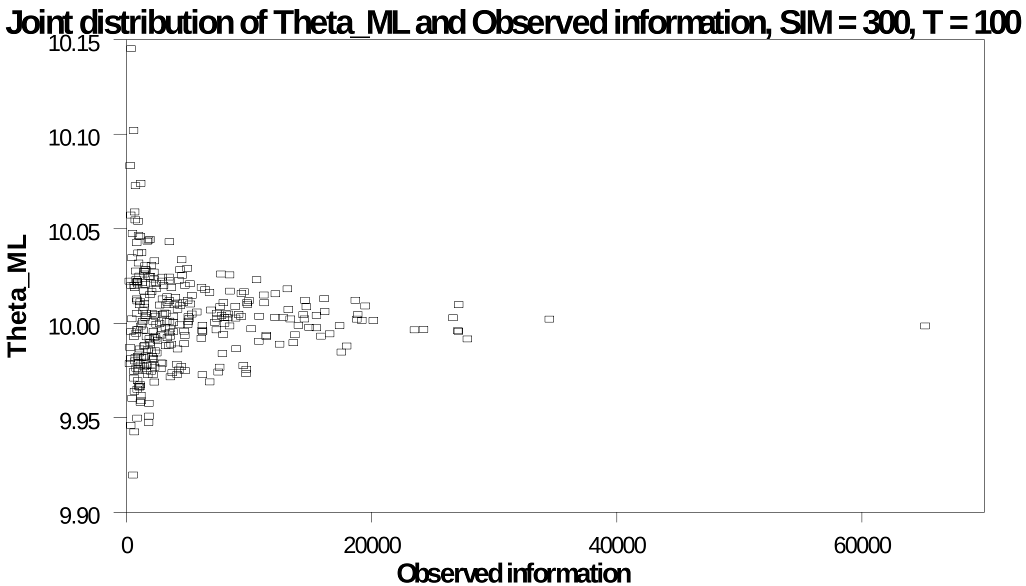
$$T(\hat{\theta} - \theta) = \frac{T^{-1} \sum_{t=1}^{T} x_{2t-1}\varepsilon_{1t}}{T^{-2} \sum_{t=1}^{T} x_{2t-1}^2} \xrightarrow{d} \frac{\int_0^1 W_2 dW_1}{\int_0^1 W_2^2(u)du} \approx \text{Mixed Gaussian}$$

For given $\{x_{2t}, t = 1, \ldots, T\}$

$$\hat{\theta}|_{\{x_{2t}\}} \sim N(\theta, \frac{\sigma_1^2}{\sum_{t=1}^{T} x_{2t-1}^2}) \text{ and } \frac{(\hat{\theta} - \theta)}{\sigma_1}(\sum_{t=1}^{T} x_{2t-1}^2)^{1/2}|_{\{x_{2t}\}} \sim N(0, 1)$$

This implies that the marginal distributions satisfy

$$\frac{\hat{\theta} - \theta}{\sqrt{Var(\hat{\theta})}} \text{ not Gaussian and } \frac{\hat{\theta} - \theta}{\sigma_1}(\sum_{t=1}^{T} x_{2t-1}^2)^{1/2} \text{ is Gaussian}$$



4.The joint distribution of $\hat{\theta}$ and the observed information $(\sum_{i=1}^{T} x_{2t-1}^2/\hat{\sigma}^2)$ in the model $x_{1t} = \theta x_{2t-1} + \varepsilon_{1t}$, and $\Delta x_{2t} = \varepsilon_{2t}$ with $\theta = 10$. Note that the larger the information, the smaller is the uncertainty in the estimate $\hat{\theta}$.

Here is an incorrect argument

1. In order to test for $\theta = \theta_0$ we apply the t-ratio $t_{\theta=\theta_0} = \frac{\hat{\theta}-\theta}{\sqrt{Var(\hat{\theta})}}$
2. To estimate the variance $Var(\hat{\theta})$ we use $Var(\hat{\theta}) \approx \frac{\hat{\sigma}_1^2}{\sum_{t=1}^T x_{2t-1}^2}$
3. Plugging in, we get $t_{\theta=\theta_0} \approx (\sum_{t=1}^T x_{2t-1}^2)^{1/2} \frac{\hat{\theta}-\theta}{\hat{\sigma}_1} \xrightarrow{d} N(0,1)$

Comments

1. is not a good idea because $Var(\hat{\theta})$ is very large and distribution of $t_{\theta=\theta_0}$ is not Gaussian
2. is incorrect because

$$T^2 Var(\hat{\theta}) = E\frac{\sigma_1^2}{T^{-2}\sum_{t=1}^T x_{2t-1}^2} \longrightarrow E\frac{\sigma_1^2}{\int_0^1 W_2^2(u)du} \neq \frac{\sigma_1^2}{\int_0^1 W_2^2(u)du} \longleftarrow \frac{\sigma_1^2}{T^{-2}\sum_{t=1}^T x_{2t-1}^2}$$

Thus 3. is not correctly argued, but the result is correct.

Always condition on the observed information $\sum_{t=1}^T x_{t-1}^2$.

Do not use expected information to estimate the variance as one usually does. Use observed information.

5. Conclusion

1. We have shown that without deterministics the process x_t behaves like a Brownian motion $CW(u)$.
2. This implies that the limit of the rank test has a generalized Dickey-Fuller distribution depending on the number of common trends as 'degrees of freedom'.
3. The limit distribution has to be modified by deterministic terms, and small sample improvements have been developed.
4. $\hat{\beta}$ is superconsistent and the asymptotic distribution is mixed Gaussian. We therefore get asymptotic χ^2 inference for tests on β.

MANY MORE TOPICS

6. Rational Expectations and the VAR
7. Seasonal cointegration
8. Models for Explosive Roots
9. The I(2) model
10. Cofractional error correction model
11. Controlling Inflation
12. Conclusion

6. Rational Expectations and the VAR

An example: Interest rates in two countries and exchange rate:
Economic expectations

$$UIP : i_t^1 - i_t^2 = E_t^{econ} \Delta e_{t+1}$$

$$E_t^{econ} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}' \begin{pmatrix} e_{t+1} \\ i_{t+1}^1 \\ i_{t+1}^2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}' \begin{pmatrix} e_t \\ i_t^1 \\ i_t^2 \end{pmatrix} \quad \text{or } E_t^{econ} c_1' x_{t+1} = c_0' x_t$$

Assume the VAR model is $\Delta x_{t+1} = \alpha\beta' x_t + \varepsilon_{t+1}$
Model based expectations $E_t x_{t+1} = (I_p + \alpha\beta')x_t$
Muth (1960): Economic expectations the same as model based expectations,

$$E_t^{econ} = E_t$$

$$E_t c_1' x_{t+1} = c_0' x_t = c_1'(I_p + \alpha\beta')x_t$$

Muth (1960): Economic expectations the same as model based expectations,

$$E_t^{econ} = E_t$$

$$E_t c_1' x_{t+1} = c_0' x_t = c_1'(I_p + \alpha\beta')x_t$$

If $r = 1$, this implies

$$c_0' = c_1'(I_p + \alpha\beta') \text{ or } c_0' - c_1' = c_1'\alpha\beta' \text{ so that } \beta = c_0 - c_1, \text{ and } c_1'\alpha = 1.$$

We therefore first test that $\beta = c_0 - c_1$, and next that $c_1'\alpha = 1$.
In the UIP example $i_t^1 - i_t^2 = E_t^{econ}\Delta e_{t+1}$ we have $\beta' x_t = i_t^1 - i_t^2$.

The equation for e_t is $\Delta e_t = \alpha_1(i_{t-1}^1 - i_{t-1}^2) + \varepsilon_{1t}$,
which implies that also $\alpha_1 = c_1'\alpha = 1$, if economic expectations are model based.

The assumption of rational (model based) expectations corresponds to specific restrictions on the reduced form coefficient from the VAR.

7. Seasonal cointegration

With roots at $z = 1, -1$, and $\Delta_1 = 1 - L$ and $\Delta_{-1} = 1 + L$, the error correction model is

$$\begin{aligned} \Delta_1 \Delta_{-1} x_t &= \frac{1}{2}\Delta_{-1}\alpha_1\beta_1' x_{t-1} - \frac{1}{2}\Delta_1 \alpha_{-1}\beta_{-1}' x_{t-1} + \varepsilon_t \\ \Pi(z) &= (1-z)(1+z)I_p - \frac{1}{2}(1+z)\alpha_1\beta_1' z - \frac{1}{2}(1+z)\alpha_{-1}\beta_{-1}' z \end{aligned}$$

with "$I(1)$ solution"

$$\begin{aligned} \Pi(z)^{-1} &= C_1 \frac{1}{1-z} + C_{-1}\frac{1}{1+z} + \sum_{i=0}^{\infty} C_i^* z^i \\ x_t &= C_1 \sum_{i=1}^{t} \varepsilon_i + C_{-1}(-1)^t \sum_{i=1}^{t} (-1)^i \varepsilon_i + A_1 + (-1)^t A_{-1} + Y_t \end{aligned}$$

Non-stationarity due to $S_t^{(1)} = \sum_{i=1}^t \varepsilon_i$ and $S_t^{(-1)} = (-1)^t \sum_{i=1}^t (-1)^i \varepsilon_i$.
Yearly aggregates of $S_t^{(-1)}$ are stationary: $\Delta_{-1} S_t^{(-1)} = (1+L)S_t^{(-1)} = \varepsilon_t$, so $\Delta_{-1}\beta_1' x_t \approx I(0)$
Yearly differences of $S_t^{(1)}$ are stationary: $\Delta_1 S_t^{(1)} = (1-L)S_t^{(1)} = \varepsilon_t$, so $\beta_{-1}'\Delta_1 x_t \approx I(0)$

8. Models for Explosive Roots

One real explosive root $\lambda < 1$, and a root at $z = 1$. Define $\Delta_\lambda = 1 - \lambda^{-1}L$. The error correction model is

$$\begin{aligned}
\Delta_1\Delta_\lambda x_t &= \frac{1}{(1-1/\lambda)}\alpha_1\beta_1'\Delta_\lambda x_{t-1} + \frac{1}{(1-\lambda)\lambda}\alpha_\lambda\beta_\lambda'\Delta_1 x_{t-1} + \varepsilon_t \\
\Pi(z) &= (1-z)(1-\lambda^{-1}z)I_p - \frac{(1-\lambda^{-1}z)}{(1-1/\lambda)}z\alpha_1\beta_1' + \frac{(1-z)}{(1-\lambda)\lambda}z\alpha_\lambda\beta_\lambda'
\end{aligned}$$

with "$I(1)$ solution"

$$\begin{aligned}
\Pi(z)^{-1} &= C_1\frac{1}{1-z} + C_\lambda\frac{1}{1-\lambda^{-1}z} + \sum_{i=0}^{\infty} C_i^* z^i \\
x_t &= C_1\sum_{i=1}^{t}\varepsilon_i + C_\lambda\lambda^{-t}\sum_{i=1}^{t}\lambda^i\varepsilon_i + \lambda^{-t}A_\lambda + A_1 + Y_t,
\end{aligned}$$

Non-stationary due to $S_t^{(1)} = \sum_{i=0}^t \varepsilon_i$, and $S_t^{(\lambda)} = \lambda^{-t}\sum_{i=0}^t \lambda^i\varepsilon_i$. The sum $\sum_{i=0}^t \lambda^i\varepsilon_i$ converges for $t \to \infty$, and the explosiveness is due to the factor $\lambda^{-t} \to \infty$. No central limit theorem involved in the limit, and asymptotic results depend on the properties ε_t.

9. The I(2) model

The error correction form of the equations

$$\begin{aligned}\Delta^2 x_t &= \alpha\beta' x_{t-1} - \Gamma\Delta x_{t-1} + \varepsilon_t,\\ \Pi(z) &= (1-z)^2 I_p - \alpha\beta' z - \Gamma(1-z)z\end{aligned}$$

with $I(2)$ solution

$$\begin{aligned}\Pi(z)^{-1} &= C_2\frac{1}{(1-z)^2} + C_1\frac{1}{(1-z)} + \sum_{i=0}^{\infty} C_i^* z^i\\ x_t &= C_2\sum_{i=1}^{t}\sum_{j=1}^{i}\varepsilon_j + C_1\sum_{i=1}^{t}\varepsilon_i + A_1 + tA_2 + Y_t\end{aligned}$$

for suitable C_1 and C_2

Δx_t is stationary, $\beta' x_t$ is $I(1)$ and there is polynomial cointegration $\beta' x_t + \delta\Delta x_t$ is $I(0)$

10. Cofractional error correction model

The vector autoregressive model for cofractional processes $(\Delta^b x_t = \sum_{i=0}^{\infty}(-1)^i\binom{b}{i}x_{t-i})$

$$\begin{aligned}
\Delta x_t &= \alpha\beta'(1-\Delta)x_t + \sum_{i=1}^{k}\Gamma_i\Delta(1-\Delta)^i x_t + \varepsilon_t \\
\Delta^b x_t &= \alpha\beta'(1-\Delta^b)x_t + \sum_{i=1}^{k}\Gamma_i\Delta^b(1-\Delta^b)^i x_t + \varepsilon_t \\
\Delta^b x_t &= \alpha\beta' L_b x_t + \sum_{i=1}^{k}\Gamma_i\Delta^b L_b^i x_t + \varepsilon_t,\ L_b = 1-\Delta^b
\end{aligned}$$

$$\begin{aligned}
\Pi(z) &= (1-z)^b I_p - \alpha\beta'(1-(1-z)^b) - \sum_{i=1}^{k}\Gamma_i(1-z)^b(1-(1-z)^b)^i \\
\Pi(z)^{-1} &= (1-z)^{-b}C + H(1-(1-z)^b) \\
x_t &= \gamma\Delta_+^{-b}\varepsilon_t + y_t^+ + \mu_t^0,\ \ y_t^+ = \sum_{n=0}^{t-1}\tau_n\varepsilon_{t-n},\ x_t\in\mathcal{F}(b),\ \beta' x_t\in\mathcal{F}(0)
\end{aligned}$$

11. Controlling Inflation

Basic assumptions

1. The central bank can determine the short rate i_t
2. Inflation appears to be nonstationary
3. The reduced form VAR model describes the behavior of agents in the sense that $E(x_t|x_{t-1})$ describes the agents plans and the outcome, x_t, deviates from $E(x_t|x_{t-1})$ by a random term ε_t

Example

Three variables: Inflation, π_t, output gap y_t and interest rate i_t and a cointegrated VAR model

$$\begin{aligned}\Delta\pi_t &= \alpha_1(i_{t-1} - \pi_{t-1} - \mu) + \varepsilon_{1t}\\ \Delta y_t &= \alpha_2(i_{t-1} - \pi_{t-1} - \mu) + \varepsilon_{2t}\\ \Delta i_t &= \alpha_3(i_{t-1} - \pi_{t-1} - \mu) + \varepsilon_{3t}\end{aligned}$$

Problem: How can the bank control inflation π_t, when only i_t can be set by the bank?

By controlling inflation we mean that we want inflation stationary around a given mean.

A suggested linear control rule (Taylor rule)

$$i_t^{ctr} = i_t + \underbrace{\kappa(\pi^* - \pi_t)}_{intervention=policy\ shock}$$

The timing of the interventions.

STEP 1: The bank investigates the economy at time t and decides to change the interest rate according to the rule

$$i_t^{ctr} = i_t + \kappa(\pi^* - \pi_t),$$

while leaving π_t and y_t intact.

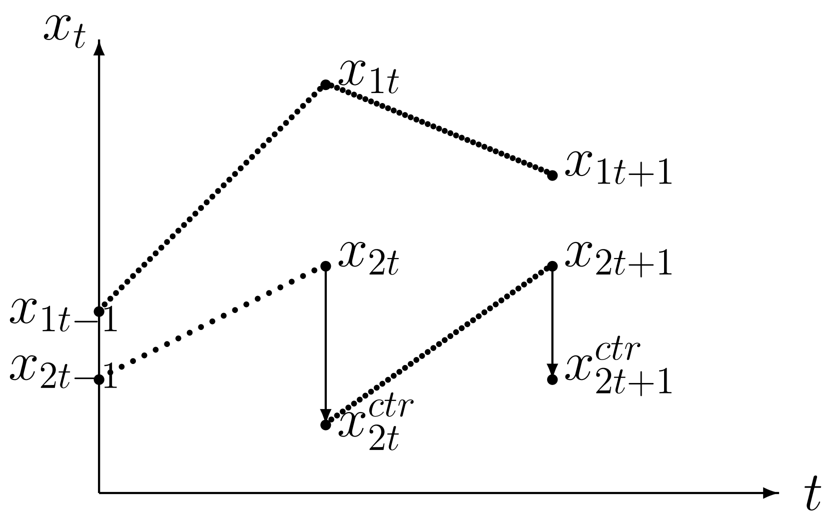
STEP 2: The market reacts to the controlled values (i_t^{ctr}, y_t, π_t) according to the VAR and creates the process x_{t+1}^{new} from the VAR which describes the plans of the agents

$$\begin{aligned} \pi_{t+1}^{new} &= \pi_t + \alpha_1(i_t^{ctr} - \pi_t - \mu) + \varepsilon_{1t+1} \\ y_{t+1}^{new} &= y_t + \alpha_2(i_t^{ctr} - \pi_t - \mu) + \varepsilon_{2t+1} \\ i_{t+1}^{new} &= i_t^{ctr} + \alpha_3(i_t^{ctr} - \pi_t - \mu) + \varepsilon_{3t+1} \end{aligned}$$

STEP1 (again): The bank investigates the economy at time $t+1$ and decides to change the interest rate according to the rule

$$i_{t+1}^{ctr} = i_{t+1}^{new} + \kappa(\pi^* - \pi_{t+1}^{new}),$$

while leaving π_{t+1}^{new} and y_{t+1}^{new} intact, etc.



1.

The process $x_t^{new} = (\pi_{t+1}^{new}, y_{t+1}^{new}, i_{t+1}^{new})'$ is observed and satisfies the equations setting $i_t^{ctr} = i_t + \kappa(\pi^* - \pi_t^{new})$:

$$\begin{aligned}
\pi_{t+1}^{new} &= \pi_t^{new} + \alpha_1(i_t^{new} + \kappa(\pi^* - \pi_t^{new}) - \pi_t^{new} - \mu) + \varepsilon_{1t+1} \\
y_{t+1}^{new} &= y_t^{new} + \alpha_2(i_t^{new} + \kappa(\pi^* - \pi_t^{new}) - \pi_t^{new} - \mu) + \varepsilon_{2t+1} \\
i_{t+1}^{new} &= i_t^{new} + \alpha_3(i_t^{new} + \kappa(\pi^* - \pi_t^{new}) - \pi_t^{new} - \mu) + \kappa(\pi^* - \pi_t^{new}) + \varepsilon_{3t+1}
\end{aligned}$$

This is a VAR with two cointegrating relations

$$\begin{aligned}
ecm1_t &= i_t^{new} + \kappa(\pi^* - \pi_t^{new}) - \pi_t^{new} - \mu \\
ecm2_t &= \pi^* - \pi_t^{new}
\end{aligned}$$

$$\begin{aligned}
\Delta\pi_{t+1}^{new} &= \alpha_1 ecm1_t + \varepsilon_{1t+1} \\
\Delta y_{t+1}^{new} &= \alpha_2 ecm1_t + \varepsilon_{2t+1} \\
\Delta i_{t+1}^{new} &= \alpha_3 ecm1_t + \kappa ecm2_t + \varepsilon_{3t+1}
\end{aligned}$$

$$\begin{aligned}\Delta\pi_{t+1}^{new} &= \alpha_1 ecm1_t + \varepsilon_{1t+1}\\ \Delta y_{t+1}^{new} &= \alpha_2 ecm1_t + \varepsilon_{2t+1}\\ \Delta i_{t+1}^{new} &= \alpha_3 ecm1_t + \kappa ecm2_t + \varepsilon_{3t+1}\end{aligned}$$

$$\alpha^{new} = \begin{pmatrix}\alpha_1 & 0\\ \alpha_2 & 0\\ \alpha_3 & \kappa\end{pmatrix}, \ \beta^{new} = \begin{pmatrix}-(1+\kappa) & -1\\ 0 & 0\\ 1 & 0\end{pmatrix},$$

Hence π_t^{new} is stationary around π^* and real interest rate still stationary around μ. Choose κ to satisfy $|eig(I_2 + \beta^{new\prime}\alpha^{new})| < 1$, to avoid explosive behavior. The dynamics of the process is changed by the repeated intervention (policy shock) of the bank. A necessary condition for controllability, in terms of the C matrix is $C_{\pi,i} \neq 0$:

$$C = \frac{1}{\alpha_3 - \alpha_1}\begin{pmatrix}\alpha_3 & 0 & -\alpha_1\\ \alpha_2 & \alpha_3 - \alpha_1 & -\alpha_2\\ \alpha_3 & 0 & -\alpha_1\end{pmatrix}$$

If x_t is $I(1)$ then the long-run impact of i_t on π_t as measured by $C_{\pi,i} = -\alpha_1/(\alpha_3-\alpha_1) \neq 0.$

12. Conclusion

This concludes the lectures on the statistical analysis of cointegration using the vector autoregressive model.

We have seen how to model the basic short- and long-run dynamics using cointegration and the error correction model, and how to formulate and test hypotheses on both cointegrating and adjustment vectors.

The algorithm of reduced rank regression replaces the ordinary least squares regression.

The asymptotic analysis is complicated by the fact that the Brownian motion and stochastic integrals enter but in reality we only use them for formulating a limit, which is then tabulated by simulation.

The model is widely used for analyzing macro data, and there are many applications and formulations waiting to discovered.