Lectures on

THE COINTEGRATED VECTOR AUTOREGRESSIVE MODEL

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LECTURE 2

The Statistical Analysis of the I(1) Model

1. The Unrestricted VAR Model
2. Estimation of the Unrestricted VAR by Regression
3. Conclusion on Analysis of Unrestricted VAR
4. Reduced Rank Regression
5. Analysis of the I(1) Cointegrated VAR Model
6. Estimation in the I(1) Model with Restrictions on β
7. Estimation in the I(1) Model with Restricted Deterministic Terms
8. Determination of Cointegration Rank
9. Conclusion 1
10. A cointegration analysis of a Dynamic Stochastic General Equilibrium model
11. Conclusion 2

1. The Unrestricted VAR Model

$$x_t = \Pi_1 x_{t-1} + \Pi_2 x_{t-2} + \Phi D_t + \varepsilon_t, \ \varepsilon_t \text{ i.i.d. } N_p(0, \Omega)$$

Parameters:

$$\theta = (\Pi_1, \Pi_2, \Phi, \Omega)$$

A hypothesis (or restricted model) is defined by a restriction

$$H : g(\theta) = 0, \text{ or } \theta = h(\phi)$$

Likelihood function defined from the conditional density

$$f_\theta(x_t | x_{t-1}, x_{t-2}, \ldots, x_0, x_{-1}) = (2\pi)^{-p/2} \frac{1}{\sqrt{\det(\Omega)}} \exp(-\frac{1}{2}\varepsilon_t' \Omega^{-1} \varepsilon_t),$$

$$\varepsilon_t = x_t - \Pi_1 x_{t-1} - \Pi_2 x_{t-2} - \Phi D_t$$

and

$$L(\theta) = \prod_{t=1}^{T} f_\theta(x_t | x_{t-1}, x_{t-2}, \ldots, x_0, x_{-1})$$

2. Estimation of the Unrestricted VAR by Regression

$$x_t = \Pi_1 x_{t-1} + \Pi_2 x_{t-2} + \Phi D_t + \varepsilon_t,\ \varepsilon_t \text{ i.i.d. } N_p(0, \Omega)$$

Let

$$B' = (\Pi_1, \Pi_2, \Phi), \quad z_t' = (x_{t-1}', x_{t-2}', D_t'), \quad x_t = B' z_t + \varepsilon_t$$

Likelihood equations and the solution

$$\sum_{t=1}^{T} (x_t - \hat{B}' z_t) z_t' = 0,\ \hat{B}' = \sum_{t=1}^{T} x_t z_t' (\sum_{t=1}^{T} z_t z_t')^{-1} = M_{xz} M_{zz}^{-1} = B' + M_{\varepsilon z} M_{zz}^{-1}$$

$$\begin{aligned} \hat{\varepsilon}_t &= x_t - \hat{B}' z_t \\ \hat{\Omega} &= T^{-1} \sum_{t=1}^{T} \hat{\varepsilon}_t \hat{\varepsilon}_t' = T^{-1}(M_{xx} - M_{xz} M_{zz}^{-1} M_{zx}) \\ L_{\max}^{-2/T} &= (2\pi e)^p |\hat{\Omega}| \end{aligned}$$

Inference is asymptotically normal:

If x_t is stationary

$$\text{Law of Large Numbers} \; : \; M_{zz} \xrightarrow{P} \Sigma > 0$$
$$\text{Central Limit Theorem} \; : \; T^{-1} M_{z\varepsilon} \xrightarrow{d} N_{p\times p}(0, \Omega \otimes \Sigma)$$

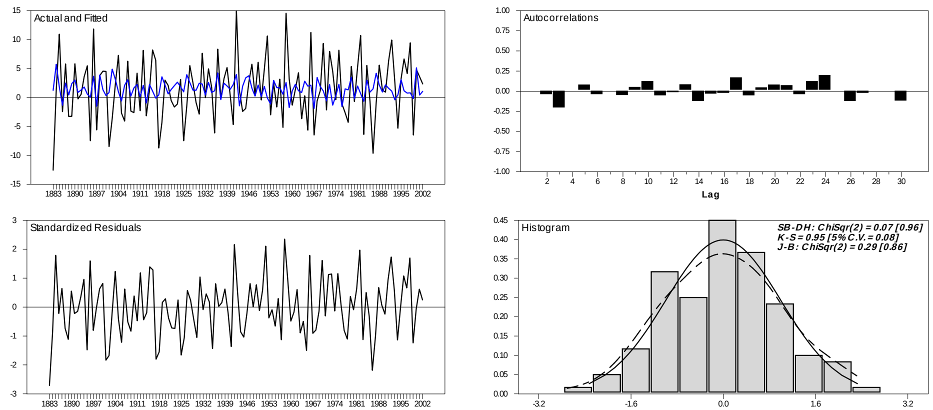
$$T^{1/2}(\hat{B} - B) \; = \; T^{1/2} M_{zz}^{-1} M_{z\varepsilon} \xrightarrow{w} N(0, \Omega \otimes \Sigma^{-1})$$

$$\hat{\Omega} \; = \; T^{-1}(M_{xx} - M_{xz} M_{zz}^{-1} M_{zx}) = T^{-1}(M_{\varepsilon\varepsilon} - M_{\varepsilon z} M_{zz}^{-1} M_{z\varepsilon}) \xrightarrow{P} \Omega$$

$$-2 \log LR(B \; = \; h(\phi)) = -2 \log \frac{\max_{B=h(\phi)} L(B)}{\max_B L(B)} \xrightarrow{w} \chi^2(f)$$

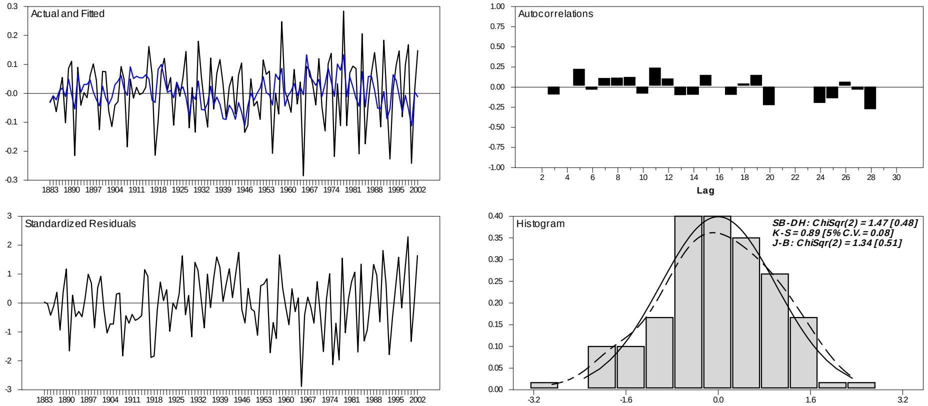
$f = \#(B) - \#(\phi)$

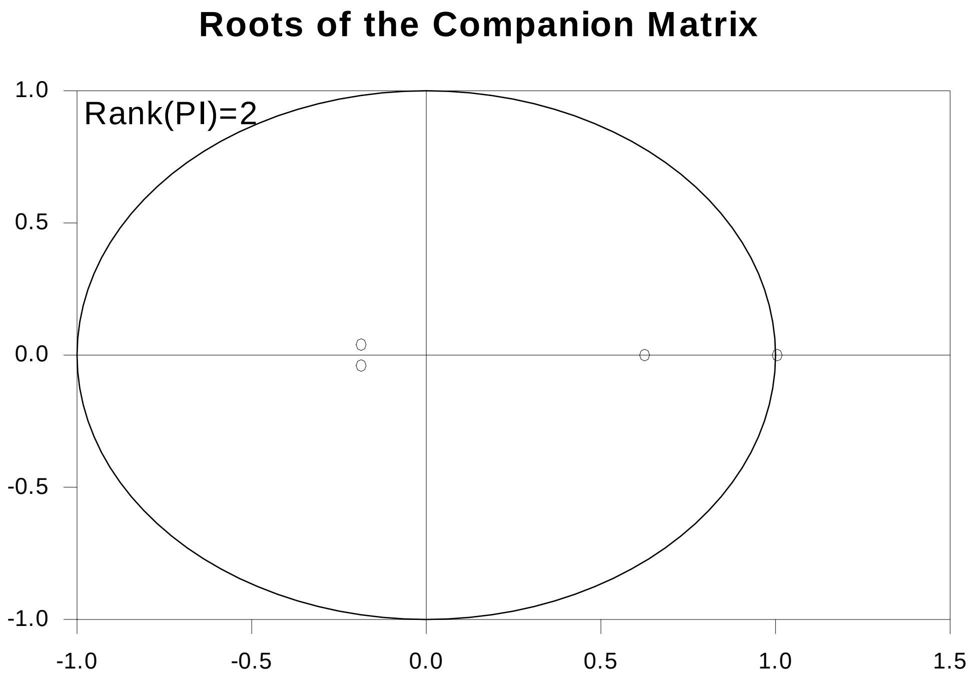
DLEVEL



1.Residual analysis.

DTEMP


Actual and Fitted
Autocorrelations
Lag
Standardized Residuals
Histogram
SB-DH: ChiSqr(2) = 1.47 [0.48]
K-S = 0.89 [5% C.V. = 0.08]
J-B: ChiSqr(2) = 1.34 [0.51]


Roots of the Companion Matrix
Rank(PI)=2
1.0
0.5
0.0
-0.5
-1.0
-1.0
-0.5
0.0
0.5
1.0
1.5

3. Conclusion on Analysis of Unrestricted VAR

Inference is based on asymptotic results. Each asymptotic result is proved under suitable assumptions, and these have to be checked for the particular application before the results can be applied. Thus the tests are interrelated and supplement each other.

The VAR model assumes

1. Linear conditional mean explained by the past observations and deterministic terms

(Check for unmodelled systematic variation, the choice of lag length, choice of information set (data), possible outliers, nonlinearity, non constant parameters)

2. Constant conditional variance

(Check for ARCH effects, but also for regime shifts in the variance)

3. Independent Normal errors, mean zero, variance Ω

(Check for lack of autocorrelation, distributional form)

Some assumptions crucial

Constant parameters, Independent errors

Others less so

ARCH, Distribution of residuals

4. Reduced Rank Regression

Consider the (non linear, reduced rank) regression

$$U_t = \alpha\beta' V_t + \Gamma Z_t + \varepsilon_t$$

where α and β are $p \times r$. Define residuals

$$\begin{aligned} R_{ut} &= U_t - M_{uz} M_{zz}^{-1} Z_t = (U_t | Z_t) \\ R_{vt} &= V_t - M_{vz} M_{zz}^{-1} Z_t = (V_t | Z_t) \end{aligned}$$

and product moments

$$\begin{pmatrix} S_{uu} & S_{uv} \\ S_{vu} & S_{vv} \end{pmatrix} = T^{-1} \sum_{t=1}^{T} \begin{pmatrix} R_{ut} \\ R_{vt} \end{pmatrix} \begin{pmatrix} R_{ut} \\ R_{vt} \end{pmatrix}'$$

The (non linear, reduced rank) regression

$$U_t = \alpha\beta' V_t + \Gamma Z_t + \varepsilon_t$$

Solve the eigenvalue problem

$$\det(\lambda S_{vv} - S_{vu} S_{uu}^{-1} S_{uv}) = 0$$

for eigenvalues and eigenvectors

$$1 > \lambda_1 > \ldots > \lambda_p > 0, \quad v_1, \ldots, v_p, \quad v_i' S_{vv} v_j = 1_{\{i=j\}}$$

The reduced rank regression estimates (T.W.Anderson 1951) are

$$\begin{aligned}
\hat{\beta} &= (v_1, \ldots, v_r) \\
\hat{\alpha} &= S_{uv}\hat{\beta}(\hat{\beta}' S_{vv}\hat{\beta})^{-1} \\
\hat{\Omega} &= S_{uu} - S_{uv}\hat{\beta}(\hat{\beta}' S_{vv}\hat{\beta})^{-1}\hat{\beta}' S_{vu}
\end{aligned}$$

This analysis will be called $RRR(U, V|Z)$

5. Analysis of the I(1) Cointegrated VAR Model

$$\mathcal{H}_r : \Delta x_t = \alpha\beta' x_{t-1} + \sum_{i=1}^{k-1} \Gamma_i \Delta x_{t-i} + \Phi D_t + \varepsilon_t,$$

where ε_t i.i.d. $N_p(0, \Omega)$ and α and β are $(p \times r)$.
$MLE = RRR(\Delta x_t, x_{t-1} | \Delta x_{t-1}, \ldots, \Delta x_{t-k+1}, D_t)$. Define

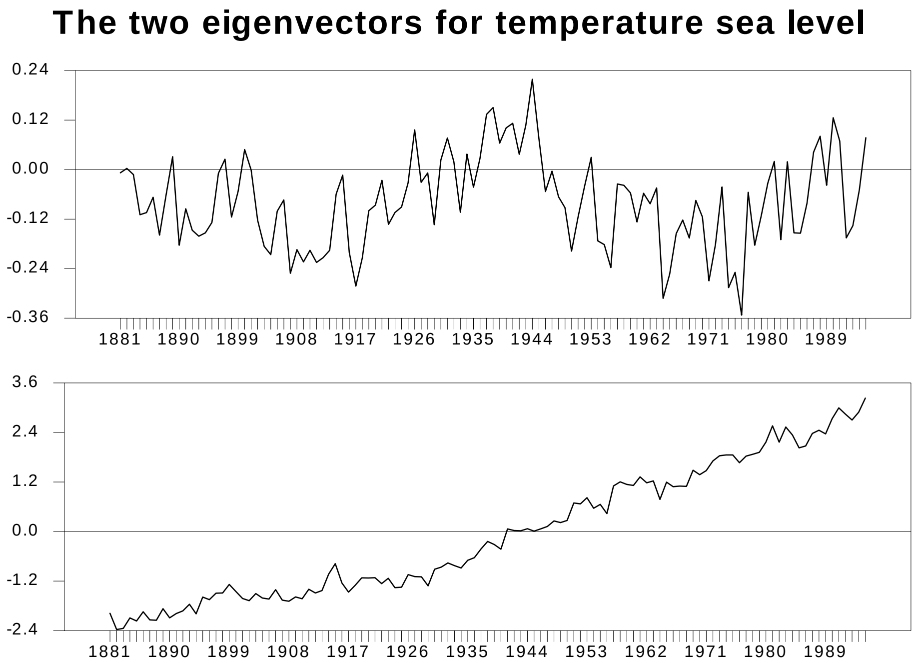
$$R_{0t} = (\Delta x_t | \Delta x_{t-1}, \ldots, \Delta x_{t-k+1}, D_t) \text{ and } R_{1t} = (x_{t-1} | \Delta x_{t-1}, \ldots, \Delta x_{t-k+1}, D_t)$$

$$S_{ij} = T^{-1} \sum_{t=1}^{T} R_{it} R'_{jt}, \quad |\lambda S_{11} - S_{10} S_{00}^{-1} S_{01}| = 0, \quad \hat{\beta} = (v_1, \ldots, v_r);$$

$$L_{\max}^{-2/T}(\mathcal{H}_r) = |S_{00}| \prod_{i=1}^{r} (1 - \hat{\lambda}_i); \; L_{\max}^{-2/T}(\mathcal{H}_p) = |S_{00}| \prod_{i=1}^{p} (1 - \hat{\lambda}_i);$$

$$-2 \log LR(\mathcal{H}_r | \mathcal{H}_p) = -T \sum_{i=r+1}^{p} \log(1 - \hat{\lambda}_i)$$

Bartlett (1938) Canonical Correlations between Δx_t and x_{t-1}.


The two eigenvectors for temperature sea level
0.24
0.12
0.00
-0.12
-0.24
-0.36
1881 1890 1899 1908 1917 1926 1935 1944 1953 1962 1971 1980 1989
3.6
2.4
1.2
0.0
-1.2
-2.4
1881 1890 1899 1908 1917 1926 1935 1944 1953 1962 1971 1980 1989

8. Determination of Cointegration Rank

The model and the test

$$\mathcal{H}_r : \Delta x_t = \alpha\beta' x_{t-1} + \sum_{i=1}^{k-1} \Gamma_i \Delta x_{t-i} + \Phi D_t + \varepsilon_t$$
$$-2\log LR(\mathcal{H}_r|\mathcal{H}_p) = -T\sum_{i=r+1}^{p} \log(1-\hat{\lambda}_i)$$

Theorem: *Under $I(1)$ assumptions and if the rank is r, the asymptotic distribution of $-2\log LR(\mathcal{H}_r|\mathcal{H}_p)$ is given as a function of Brownian motion of dimension $p-r$. The distribution depends on the type of deterministic terms and is tabulated by simulation.*

We first test

$$\mathcal{H}_0 : r = 0 \text{ versus } \mathcal{H}_p : r \leq p, \quad \text{Test: } -2\log LR(\mathcal{H}_0|\mathcal{H}_p) = -T\sum_{i=1}^{p} \log(1-\hat{\lambda}_i).$$

If this is rejected, we test

$$\mathcal{H}_1 : r = 1 \text{ versus } \mathcal{H}_p : r \leq p, \quad \text{Test: } -2\log LR(\mathcal{H}_1|\mathcal{H}_p) = -T\sum_{i=2}^{p} \log(1-\hat{\lambda}_i) \text{ etc.}$$

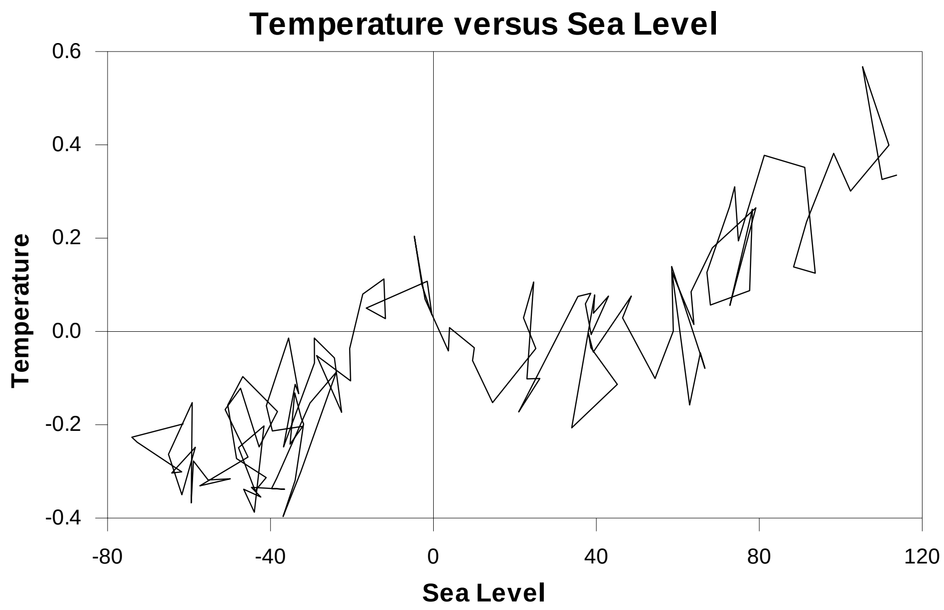
Rank determination for temperature and sea level data					
p-r	r	EigVal	TraceTest	95Fract	p-val
2	0	0.168	20.76	15.41	0.005
1	1	0.003	0.36	3.84	0.54

The fitted ECM model for temperature and sea level data

$$\Delta h_t = \underset{(t=0.86)}{4.15}\,(T_{t-1} - \underset{(t=-7.37)}{0.0031}\,h_{t-1}) - \underset{(t=-3.11)}{0.2805}\,\Delta h_{t-1} + \underset{(t=0.60)}{3.04}\,\Delta T_{t-1} + \underset{(t=3.55)}{2.22}$$

$$\Delta T_t = \underset{(t=-4.26)}{-0.40}\,(T_{t-1} - \underset{(t=-7.37)}{0.0031}\,h_{t-1}) - \underset{(t=-1.40)}{0.0024}\,\Delta h_{t-1} - \underset{(t=-0.54)}{0.053}\,\Delta T_{t-1} - \underset{(t=-1.91)}{0.023}$$

Note h_t weakly and strongly exogenous

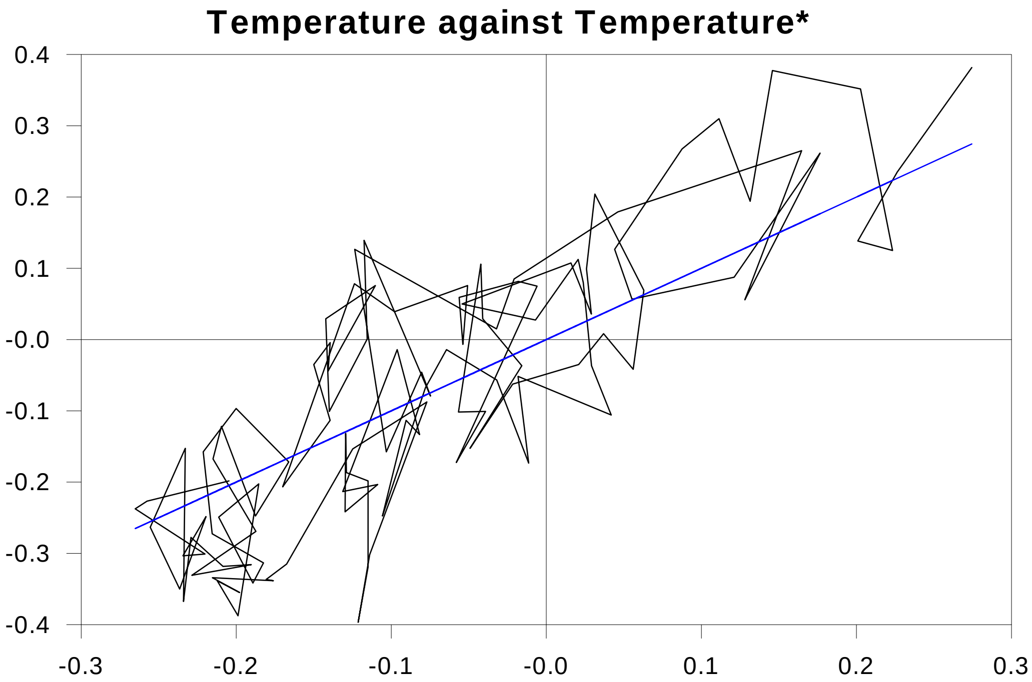

Temperature versus Sea Level
0.6
0.4
0.2
0.0
-0.2
-0.4
Temperature
-80
-40
0
40
80
120
Sea Level

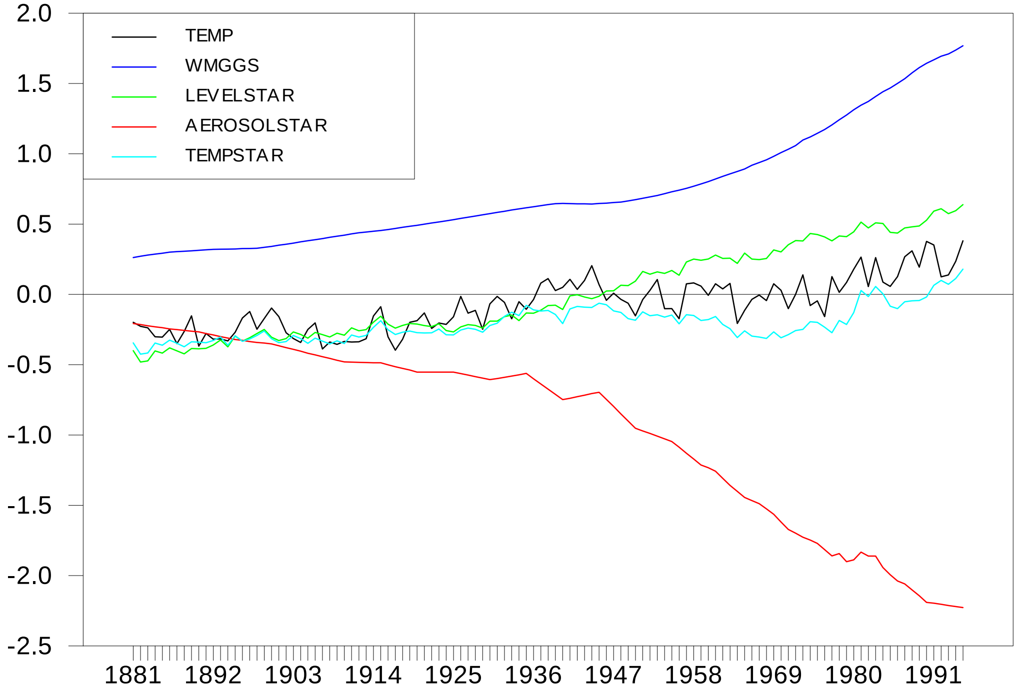
A partial model for (T_t, h_t) conditional on the forcing (weakly exogenous) variables $WMGG$ $(CO2$ and $Methan)$ and $Aerosols$ ($Sulphate$)

$$\Delta \begin{pmatrix} T_t \\ h_t \end{pmatrix} = \alpha \left(\beta_1' \begin{pmatrix} T_{t-1} \\ h_{t-1} \end{pmatrix} + \beta_2' \begin{pmatrix} WMGG_{t-1} \\ Aerosol_{t-1} \end{pmatrix} \right) + \cdots + \varepsilon_t$$

Cointegrating relation

$$\hat{\beta}' x_t = T_t - \underset{(t=-3.52)}{0.0065}\, h_t - \underset{(t=4.68)}{0.768}\, wmgg_t - \underset{(t=3.51)}{1.478}\, aerosol_t$$


Temperature against Temperature*
0.4
0.3
0.2
0.1
-0.0
-0.1
-0.2
-0.3
-0.4
-0.3
-0.2
-0.1
-0.0
0.1
0.2
0.3



2.Plot of the components of the fitted Temperature for simplified forcing variables

6. Estimation in the I(1) Model with Restrictions on β

Consider the model

$$\Delta x_t = \alpha\beta' x_{t-1} + \sum_{i=1}^{k-1} \Gamma_i \Delta x_{t-i} + \Phi D_t + \varepsilon_t,$$

and the hypothesis $\mathcal{H} : \beta = H\phi$, so that under $\mathcal{H}$

$$\Delta x_t = \alpha\phi' H' x_{t-1} + \sum_{i=1}^{k-1} \Gamma_i \Delta x_{t-i} + \Phi D_t + \varepsilon_t,$$

Estimate by $RRR(\Delta x_t, H' x_{t-1} | \Delta x_{t-1}, \ldots, \Delta x_{t-k+1}, D_t)$.
Similarly for hypotheses $\beta = (b, H\phi), \alpha = H\psi, \alpha_\perp = H\psi$

But not for $\beta = (H_1\phi_1, H_2\phi_2)$ and general non-linear hypotheses.
General optimization algorithm or switching algorithms used for programs.

7. Estimation in the I(1) Model with Restricted Deterministic Terms

The model for α and β $(p \times r)$ with restricted constant term $\mu_1 = 0,\ \alpha'_\perp \mu_0 = 0$

$$\Delta x_t = \alpha\beta' x_{t-1} + \sum_{i=1}^{k-1} \Gamma_i \Delta x_{t-i} + \alpha\beta_0' + \varepsilon_t = \alpha \begin{pmatrix} \beta \\ \beta_0 \end{pmatrix}' \begin{pmatrix} x_{t-1} \\ 1 \end{pmatrix} + \sum_{i=1}^{k-1} \Gamma_i \Delta x_{t-i} + \varepsilon_t$$

Estimate by $RRR(\Delta x_t, \begin{pmatrix} x_{t-1} \\ 1 \end{pmatrix} | \Delta x_{t-1}, \ldots, \Delta x_{t-k+1})$

The model for α and β $(p \times r)$ with restricted linear term $\alpha'_\perp \mu_1 = 0$

$$\begin{aligned} \Delta x_t &= \alpha\beta' x_{t-1} + \sum_{i=1}^{k-1} \Gamma_i \Delta x_{t-i} + \mu_0 + \alpha\beta_1' t + \varepsilon_t, \\ &= \alpha \begin{pmatrix} \beta \\ \beta_1 \end{pmatrix}' \begin{pmatrix} x_{t-1} \\ t \end{pmatrix} + \sum_{i=1}^{k-1} \Gamma_i \Delta x_{t-i} + \mu_0 + \varepsilon_t \end{aligned}$$

Estimate by $RRR(\Delta x_t, \begin{pmatrix} x_{t-1} \\ t \end{pmatrix} | \Delta x_{t-1}, \ldots, \Delta x_{t-k+1}, 1)$

9. Conclusion

We have shown that the unrestricted VAR model is estimated by Ordinary Least Squares, but the cointegrated VAR model is estimated by Reduced Rank Regression.

The same algorithm can be used for a number of useful hypotheses on the cointegrating coefficients and adjustment parameters. The algorithm can also be used for suitable restrictions on the deterministic terms.

The Reduced Rank Regression solves all the models $\mathcal{H}_0, \ldots, \mathcal{H}_p$ simultaneously.

10. A cointegration analysis of a Dynamic Stochastic General Equilibrium model

This analysis of a DSGE model is based upon

1. Ireland, P. 2004. A method for taking models to the data. *Journal of Economic Dynamics and Control* 28, 1205-1226.

2. Juselius, K. and Franchi, M. (2007) Taking a DSGE Model to the Data Meaningfully. *Economics: The Open-Access, Open-Assessment E-Journal,* 1, 2007-4.

The Data

The data 1948:1 to 2002:2 from
Federal Reserve Bank of St. Louis' FRED database and
Bureau of Labor Statistics' Establishment Survey.

$N_t =$ Civilian, non-institutional population, age 16 and over.
$C_t =$ Real Personal Consumption Expenditures in chained 1996 dollars/N_t
$I_t =$ Real Gross Private Domestic Investment in chained 1996 dollars$/N_t$
$H_t =$ Hours of wage and salary workers on private, non-farm payrolls$/N_t$.
$Y_t = I_t + C_t$

THE DSGE MODEL

The Cobb-Douglas production function is $Y_t = A_t K_t^\theta (\eta^t H_t)^{1-\theta}$ where η is 'gross rate of labor-augmenting technological progress'.

Utility function $E_t[\sum_{i=0}^{\infty} \beta^i (\log C_{t+i} - \gamma H_{t+i})]$ is maximized with respect to $\{C_t, H_t\}_{t=0}^{\infty}$ subject to

$$\begin{aligned} \log(A_t) &= (1-\rho)\log A + \rho \log(A_{t-1}) + \varepsilon_t, \quad |\rho| < 1 \\ Y_t &= C_t + I_t \\ K_{t+1} &= (1-\delta)K_t + I_t \end{aligned}$$

First order conditions

$$\begin{aligned} \gamma C_t H_t &= (1-\theta)Y_t \\ 1 &= \beta E_t[\frac{C_t}{C_{t+1}}(\frac{\theta Y_{t+1}}{K_{t+1}} + 1 - \delta)] \end{aligned}$$

Statistical Assumptions and Analysis of Peter Ireland

The variables $(y_t = \log Y_t, k_t = \log K_t, i_t = \log I_t, c_t = \log C_t)$ trend stationary (with same trend):

$$\begin{aligned}
\hat{y}_t &= \log Y_t - t\log\eta - y, \quad \hat{k}_t = \log K_t - t\log\eta - k \\
\hat{i}_t &= \log I_t - t\log\eta - i, \quad \hat{c}_t = \log C_t - t\log\eta - c \\
\hat{a}_t &= \log A_t - a, \quad \hat{h}_t = \log H_t - h
\end{aligned}$$

stationary mean zero. Steady state values are (y, k, i, c, a, h).
Linearized first order conditions

$$\begin{aligned}
FOC1 &: \gamma C_t H_t = (1-\theta)Y_t : c_t + h_t = y_t + \gamma \\
FOC2 &: 1 = \beta E_t[\frac{C_t}{C_{t+1}}(\frac{\theta Y_{t+1}}{K_{t+1}} + 1 - \delta)] : E_t\Delta c_{t+1} = \gamma_1 + \gamma_2 E_t(y_{t+1} - k_{t+1}) \\
Cobb-Douglas &: Y_t = A_t K_t^{\theta}(\eta^t H_t)^{1-\theta} : y_t = a_t + \theta k_t + (1-\theta)h_t + (1-\theta)t\log\eta
\end{aligned}$$

The linearized economic theory model in state space form, because k_t is unobserved

$$\text{state equation}: \begin{pmatrix} \hat{k}_t \\ \hat{a}_t \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ 0 & \rho \end{pmatrix} \begin{pmatrix} \hat{k}_{t-1} \\ \hat{a}_{t-1} \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \varepsilon_t$$

$$\text{observation equation}: \begin{pmatrix} \hat{y}_t \\ \hat{c}_t \\ \hat{h}_t \end{pmatrix} = C \begin{pmatrix} \hat{k}_t \\ \hat{a}_t \end{pmatrix}$$

The steady state values and (a_1, a_2, C) are (simple) computable functions of parameters: $\tau = (\beta, \gamma, \theta, \eta, \delta, A, \rho, \sigma)$. (http://www2.bc.edu/~irelandp.)

Observation equation is singular

‘We need a stochastic formulation to make simplified relations elastic enough for applications’, Haavelmo (1943).

The model considered by Ireland adds an autoregressive error to the observation equation

$$\hat{k}_t = a_1 \hat{k}_{t-1} + a_2 \hat{a}_{t-1}$$
$$\hat{a}_t = \rho \hat{a}_{t-1} + \varepsilon_t$$

$$\begin{pmatrix} \hat{y}_t \\ \hat{c}_t \\ \hat{h}_t \end{pmatrix} = C \begin{pmatrix} \hat{k}_t \\ \hat{a}_t \end{pmatrix} + u_t$$
$$u_t = D u_{t-1} + \xi_t$$

ξ_t, i.i.d. $N_3(0, V)$ independent of ε_t i.i.d. $N(0, \sigma^2)$.

Thus the **five** variables, $\hat{y}_t, \hat{k}_t, \hat{c}_t, \hat{h}_t, \hat{a}_t$ are driven by **four** errors, $\varepsilon_t, \xi_{1t}, \xi_{2t}, \xi_{3t}$.

THE VAR APPROACH.

The Data

The data 1948:1 to 2002:2 from
Federal Reserve Bank of St. Louis' FRED database and
Bureau of Labor Statistics' Establishment Survey.
$N_t =$ Civilian, non-institutional population, age 16 and over.
$C_t =$ Real Personal Consumption Expenditures in chained 1996 dollars/N_t
$I_t =$ Real Gross Private Domestic Investment in chained 1996 dollars$/N_t$
$H_t =$ Hours of wage and salary workers on private, non-farm payrolls$/N_t$.
$Y_t = I_t + C_t$
add the variable
$K_t =$ Capital Stock Formation$/N_t$
$x_t = (\log Y_t, \log C_t, \log K_t, \log H_t)$
Analyse the data in two periods $1960:1-1979:4$ and $1981:1-2002:1$
Model in each period

$$\Delta x_t = \alpha\beta' x_{t-1} + \Gamma_1 \Delta x_{t-1} + \mu_0 + \alpha\gamma_1 t + \Phi D_t + \varepsilon_t$$

Hypotheses of interest: two scenarios

Possibility I: If $\log A_t$ is non stationary then

$$
\begin{aligned}
I(1) &\approx y_t - \theta k_t - (1-\theta)h_t \;\; \text{(unit root in } a_t) \\
I(0) &\approx c_t - y_t + h_t \quad (FOC1) \\
I(0) &\approx y_t - k_t \quad (FOC2) \\
I(0) &\approx h_t
\end{aligned}
$$

Possibility II: If $\log A_t$ is stationary then

$$
\begin{aligned}
I(0) &\approx y_t - \theta k_t - (1-\theta)h_t \\
I(0) &\approx c_t - y_t + h_t \quad (FOC1) \\
I(0) &\approx y_t - k_t \quad (FOC2) \\
I(1) &\approx h_t
\end{aligned}
$$

Main assumptions in Irelands paper

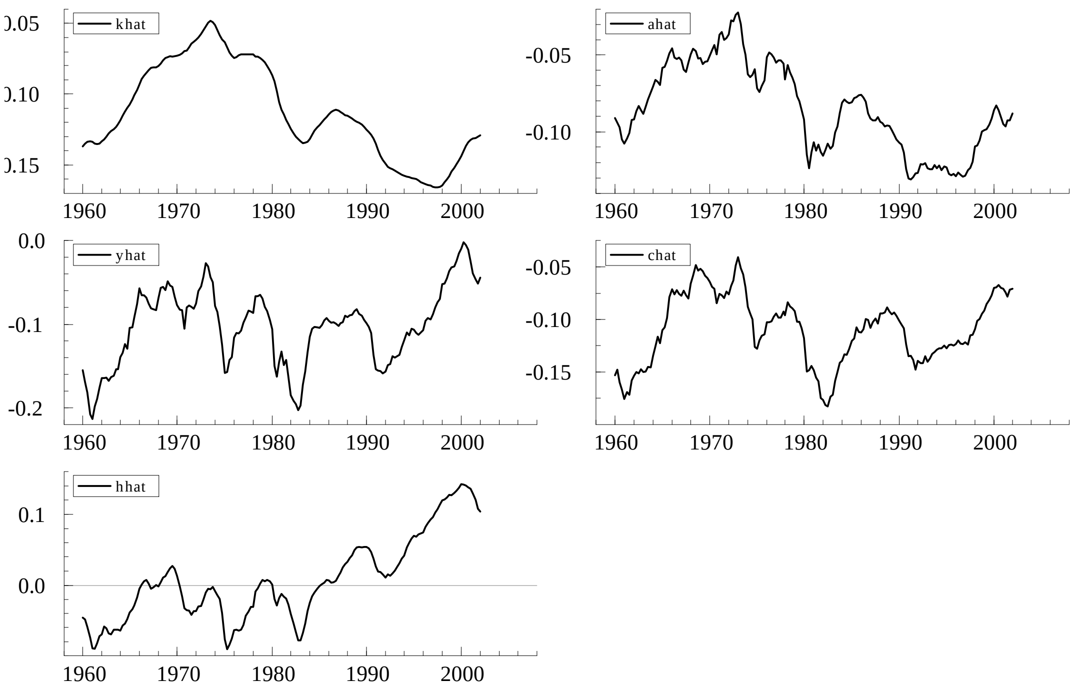
- Strutural assumptions: Parameters constant over time
- Exogeneity assumptions: $\log A_t$ and $\log K_t$ drive the system (technology shocks)
- Stationarity assumptions $\log Y_t, \log K_t, \log C_t, \log I_t$ trend stationary, $\log A_t, \log H_t$ stationary, the errors u_t are stationary
- Distributions ε_t, ξ_t Gaussian and $\{\varepsilon_t\}$ independent of $\{\xi_t\}$

Results

Coeff	Estimate	error	$\underset{p-val}{Autocorr}$	$\underset{p-val}{ARCH}$	$\underset{p-val}{Normal}$
ρ	0.9987	ε_t	0.004	0.78	0.00
a_1	0.8824	ξ_1	0.188	0.03	0.10
a_2	0.1568	ξ_2	0.001	0.00	0.10
$\rho_{\max}(D)$	0.9398	ξ_3	0.010	0.00	0.00

A break in the parameters was observed by Peter Ireland but ignored in the analysis

Conclusion Statistical evidence from an analysis of the DSGE model cannot be consider reliable.


khat
).05
).10
).15
ahat
-0.05
-0.10
yhat
0.0
-0.1
-0.2
chat
-0.05
-0.10
-0.15
hhat
0.1
0.0
1960
1970
1980
1990
2000

Determination of cointegration rank for two periods							
		1960:1-1979:4			1981:1-2002.1		
r	$p-r$	λ	$p-val$	$\rho_{\max}$	λ	$p-val$	$\rho_{\max}$
0	4	0.47	0.01	0.56	0.51	0.00	0.77
1	3	0.25	0.43	0.73	0.27	0.03	0.71
2	2	0.13	0.76	0.77	0.20	0.26	0.81
3	1	0.09	0.58	0.90≈ 1	0.08	0.44	0.96≈ 1
4	0			0.94≈ 1			0.98≈ 1

We take the models for period I and II with two lags and rank = 2 and test

- h_t stationary (unit vector in β), I: Reject (0.01), II: Reject (0.00)
- y_t, c_t, k_t are trend stationary (unit vector in β), I: Reject, II: Reject (all 0.00)
- a_t, k_t acts as the main driving forces in model, i.e. k_t weakly exogenous, I: Reject (0.00), II: Reject (0.00)

However,

- c_t is weakly exogenous in both periods (0.15 and 0.41) and h_t in peridod II (0.07)

(Thus demand shocks (labour shocks) rather than supply shocks drive the economy)

The shocks to k_t do not contribute to the trends; a unit vector in α, I: Accept (0.62), II: Accept (0.65).

More tests

- a_t is stationary, $y_t - \theta k_t - (1-\theta)h_t - t \log \eta$ stationary,I: Accept, II: Accept

$$\begin{array}{cc} 1960:1-1979:4 & 1981:1-2002.1 \\ y_t = 0.65h_t + 0.35k_t \ (0.30) & y_t = 0.39h_t + 0.61k_t \ (0.17) \end{array}$$

- $y_t - c_t$ trend stationary I: Reject (0.03), II: Reject (0.00)
- $(FOC1)$ $y_t - c_t - h_t$ trend stationary,I: Reject (0.00), II: Reject (0.02)
- $(FOC2)$ $y_t - k_t$, trend stationary,I: Reject (0.00), II: Reject (0.046)

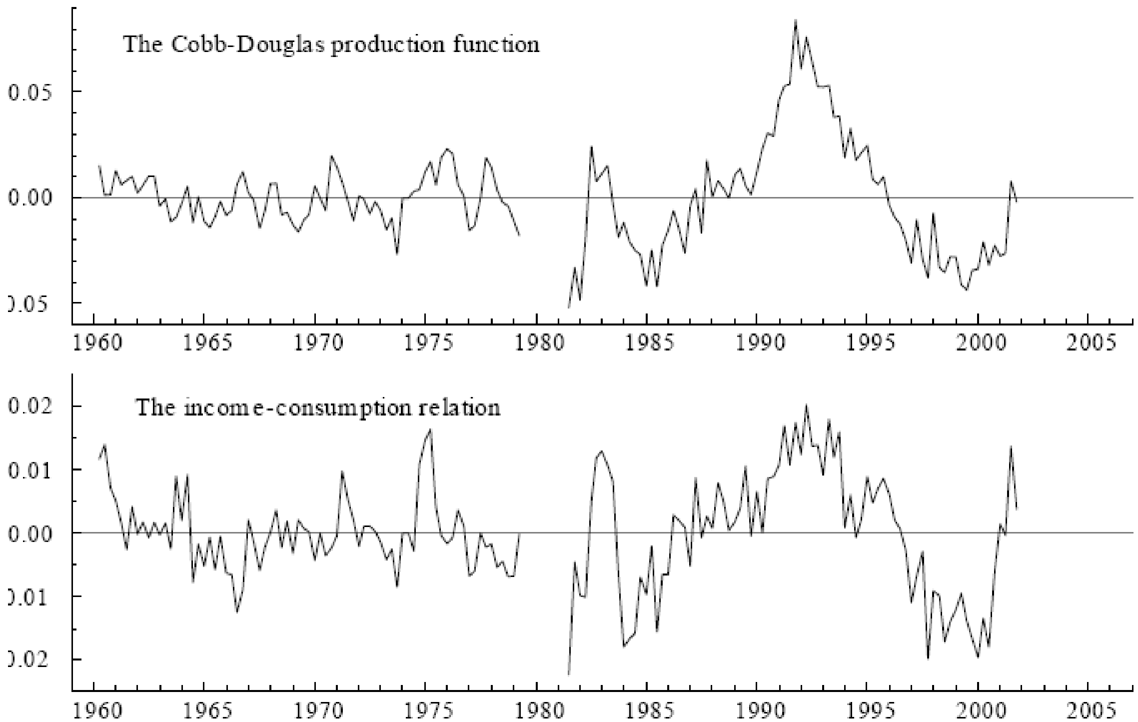


Figure 7: The Cobb-Douglas function and the income-consumption ratio relation estimated over the two regimes.

11. Conclusion 2

When taking a model to the data one shoudl carefully check the assumptions behind the model one applies for inference.

If we find a model that describes the data well, we can test the validity of the economic model assumptions, and if they are accepted we can test hypotheses with in the model.

If they are rejected we can perhaps gain so much information on the behaviour of the data that we can formulate a new model and there by gain insight into the functioning of the economy.