

# ROBUST INFERENCE IN NON-STATIONARY TIME SERIES

THESIS Submitted in fulfillment of the requirements for the degree of Doctor of Science in Economics in

**Higher School of Economics**  
**Pre-defence in Gaidar institute**



ИНСТИТУТ  
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# Motivation

- **Unit roots**: do the data support the view that the trend is changing every period or never?
- **Structural breaks**: may be trend is changing sometimes? Changes in persistence?
- **Cointegration**: univariate and multivariate; with or without breaksю
- Inference on the **break dates**
- **Predictability** with non-stationary regressors
- **Non-stationary volatility**: breaks in volatility, regime switching, etc.
- **Explosive bubbles**: identificatio and testing



# Motivation

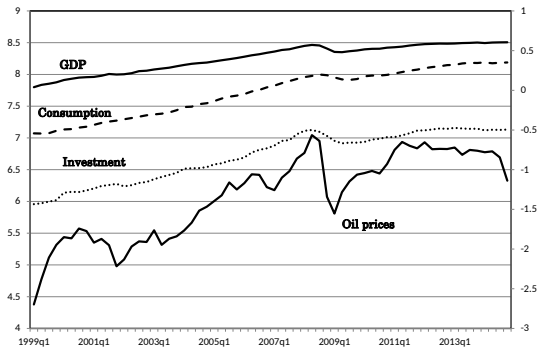


Figure 1: Russian macroeconomic series and oil prices



# Testing for a unit root with possible presence of structural breaks

- Influence of linear **trend and/or the initial condition** on unit root testing (Harvey et al. (2009), Harvey et al. (2012a))
- Stationarity testing (Muller (2005), Harris et al. (2007))
- Uncertainty over the break in unit root testing (Harvey et al. (2012b), Harvey et al. (2013b))
- **Invalidity of Zivot and Andrews (1992)** (Harvey et al. (2013a))
- Robust testing for the trend function under uncertainty whether the series is stationary,  $I(0)$ , or integrated of order one,  $I(1)$



# Outline

- 1 Motivation
- 2 Testing for a unit root with possible presence of structural breaks
  - Trend and initial condition in stationarity tests
    - Pre-testing for the trend
    - On trend breaks and initial condition in unit root testing
- 3 Bootstrap inference for non-stationary time series
  - Testing for change in persistence
  - Testing for seasonal non-stationarity under time-varying volatility
- 4 Inference on structural breaks in univariate cointegrated models
- 5 Testing for predictability with possibly non-stationary and endogenous predictors under non-stationary volatility
- 6 Testing and dating for explosive bubbles
  - Identification and dating the (explosive) bubbles under non-stationary volatility
  - Dating the bubble
- 7 Reviews
- 8 Empirics



# Trend and initial condition in stationarity tests

- (Skrobotov, 2015):

$$y_t = \mu + \beta t + u_t, \quad t = 1, \dots, T, \quad (1)$$

$$u_t = \rho u_{t-1} + \varepsilon_t, \quad t = 2, \dots, T, \quad \rho = \rho_T = 1 - c/T \quad (2)$$

- Null of stationarity (local to unit root)  $H_0 : c \geq \bar{c} > 0$  against  $H_1 : c = 0$
- Combination of  $Q(\bar{c})$  tests of Muller (2005) and  $S(\bar{c})$  tests of Harris et al. (2007)

## Definition 1

The modified intersection of rejections strategy  $IR_4^*$  is defined as follows:

- 1) If  $s_\beta \leq cv_\beta$  and  $s_\alpha \leq cv_\alpha$ , then use the liberal decision rule  $IR(Q^\mu, Q^\tau, S^\mu, S^\tau)$ ;
- 2) If  $s_\beta \leq cv_\beta$  and  $s_\alpha > cv_\alpha$ , then use the liberal decision rule  $IR(S^\mu, S^\tau)$ , the corresponding scaling constant is defined as  $m_\xi^{**}$ ;
- 3) If  $s_\beta > cv_\beta$  and  $s_\alpha \leq cv_\alpha$ , then use the liberal decision rule  $IR(Q^\tau, S^\tau)$ , the corresponding scaling constant is defined as  $m_\xi^\tau$ ;
- 4) If  $s_\beta > cv_\beta$  and  $s_\alpha > cv_\alpha$ , then use the decision rule reject  $H_0$ , if  $S^\tau > cv_\xi^{S, \tau}$ .

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# Pre-testing for the trend

- Skrobotov (2022b):  $H_0 : \beta = 0$

$$I(0) : \quad T^{3/2}(\hat{\beta} - \beta) \rightarrow_d 12\omega_u \int_0^1 (s - 1/2)dB(r) = N(0, 12\omega_u^2),$$

$$I(1) : \quad T^{1/2}(\hat{\beta} - \beta) = \frac{T^{-5/2} \sum_{t=1}^T tu_t}{T^{-3} \sum_{t=1}^T t^2} \rightarrow_d 3\omega_u \int_0^1 rB(r)dr = N\left(0, \frac{6}{5}\omega_u^2\right),$$

- Ibragimov and Müller (2010): **partition of the data into some fixed number  $q \geq 2$  of equal groups of consecutive observations**, consider group estimators  $\hat{\beta}_j, j = 1, \dots, q, T^\delta(\hat{\beta}_1, \dots, \hat{\beta}_q) \rightarrow_d N(0, \Sigma)$  with some diagonal covariance matrix  $\Sigma$  and  $\delta = 1/2$  or  $3/2$  depending on the order of integration of  $u_t$ .

$$t_\beta = \sqrt{q} \frac{\bar{\hat{\beta}} - \beta_0}{s_{\hat{\beta}}} \quad (3)$$

with  $\bar{\hat{\beta}} = q^{-1} \sum_{j=1}^q \hat{\beta}_j$  and  $s_{\hat{\beta}}^2 = (q-1)^{-1} \sum_{j=1}^q (\hat{\beta}_j - \bar{\hat{\beta}})^2$

- Null hypothesis  $H_0$  is rejected at level  $\alpha \leq 0.05$  if  $|t_\beta|$  exceeds the  $(\alpha/2)$  percentile of the Student's  $t$ -distribution with  $q-1$  degrees of freedom.





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# On trend breaks and initial condition in unit root testing

- Skrobotov (2018b): extension of Harvey et al. (2012a)

$$y_t = \mu + \beta t + \gamma_T DT_t(\lambda_0) + u_t, \quad t = 1, \dots, T, \quad (4)$$

$$u_t = \rho_T u_{t-1} + \varepsilon_t, \quad t = 2, \dots, T, \quad (5)$$

where  $DT_t(\lambda_0) = (t - \lfloor \lambda_0 T \rfloor) \mathbb{I}(t > \lfloor \lambda_0 T \rfloor)$ ,  $\mathbb{I}(\cdot)$  is the indicator function and the trend break occurs at time  $\lfloor \lambda_0 T \rfloor$  (where  $\lambda_0$  is the corresponding break fraction).

- For small initial conditions, the minimum GLS-based test *MDF-GLS* by Harvey et al. (2013b) should be used, and for large initial conditions, the combination of minimum OLS-based test *MDF-OLS* (this is Zivot and Andrews (1992) test) and OLS-based test with the estimated break date *ADF-OLS<sup>tb</sup>*( $\hat{\lambda}^{D_m}$ ) (where the break fraction  $\hat{\lambda}^{D_m}$  is estimated as in (Harvey and Leybourne, 2013)) should be used, dependent on the robust tests for breaks.
- The modified  $A^*(s_\kappa, s_\alpha)$  strategy is proposed.



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# Testing for change in persistence

$$y_t = \mu + \beta t + \rho_t y_{t-1} + u_t, \quad (6)$$

- $\rho_t = 1$  ( $y_t \sim I(1)$ ) for  $t = 1, \dots, T_1$   
 $\rho_t < 1$  ( $y_t \sim I(0)$ ) for  $t = T_1 + 1, \dots, T$
- or vice versa (Kim, 2000; Kurozumi, 2005; Kejriwal and Perron, 2012)

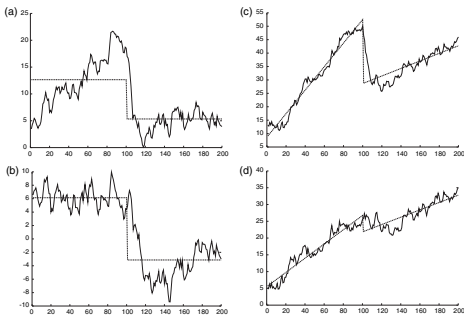


Figure 1. The simulated series; (a) non-trending case:  $\alpha = 1 \rightarrow 0.85$ , (b) non-trending case:  $\alpha = 0.85 \rightarrow 1$ , (c) trending case:  $\alpha = 1 \rightarrow 0.85$ , (d) trending case:  $\alpha = 0.85 \rightarrow 1$



# Testing for change in persistence

- Skrobotov (2018a, 2022a): **likelihood ratio test** for the null hypothesis  $I(1)$  against a change in persistence,  $I(0) \rightarrow I(1)$  or  $I(1) \rightarrow I(0)$

$$y_t = \beta' d_t + u_t, \quad t = 1, \dots, T, \quad (7)$$

$$u_t = \rho_t u_{t-1} + \varepsilon_t, \quad t = 2, \dots, T, \quad (8)$$

- Rewrite:

$$y_t = \rho_1 D_1 y_{t-1} + \rho_2 D_2 y_{t-1} + \varepsilon_t, \quad (9)$$

where  $D_1 = \mathbb{I}(t \leq \lfloor \lambda T \rfloor)$ ,  $D_2 = 1 - D_1$ .

- Maximization of the likelihood over  $\bar{\rho}_1$  and  $\bar{\rho}_2$  (“d” means detrended):

$$LR(\lambda) = \max_{\bar{\rho}_1 \leq 1, \bar{\rho}_2 \leq 1} \mathcal{L}^d(\bar{\rho}_1, \bar{\rho}_2, \lambda; \hat{\sigma}^2, \hat{\phi}^2) - \mathcal{L}^d(1, 1, \lambda; \hat{\sigma}^2, \hat{\phi}^2), \quad (10)$$

- Unknown date of change (exp-type test), lag length selection, sieve bootstrap



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# Testing for seasonal non-stationarity under time-varying volatility

Cavaliere, Skrobotov and Taylor (2019)

- Extension of Cavaliere and Taylor (2008a) to the seasonal context.
- Adopt a periodic non-stationary volatility set-up which includes both the form of Periodic Heteroskedasticity considered in Burrige and Taylor (2001).
- Univariate seasonal time series  $\{x_{Sn+s}\}$ :

$$\alpha(L)x_{Sn+s} = u_{Sn+s}, \quad s = 1 - S, \dots, 0, \quad n = 2, \dots, N, \quad (11)$$

$$\phi(L)u_{Sn+s} = \varepsilon_{Sn+s} \quad (12)$$

$$\varepsilon_{Sn+s} = \sigma_{Sn+s}e_{Sn+s} \quad (13)$$

where  $S$  – number of seasons,  $\alpha(L) = 1 - \sum_{j=1}^S \alpha_j L^j$  is an  $S$ -order autoregressive polynomial,  $\phi(L) = 1 - \sum_{j=1}^p \phi_j L^j$  is a  $p$ th order autoregressive polynomial,  $L$  is the lag operator such that  $L^{Sj+k}y_{Sn+s} = y_{S(n-j)+s-k}$ . Sample size is  $T := SN$ ,  $N$  – number of seasonal cycles (eg years).



# Testing for seasonal non-stationarity under time-varying volatility

- Test for seasonal unit roots in  $\alpha(L)$  polynomial
- $S$ th order polynomial  $a(L)$  can be factorised as  $\alpha(L) = \prod_{k=0}^{\lfloor S/2 \rfloor} \omega_k(L)$
- $\omega_0(L) := (1 - \alpha_0 L)$  associates the parameter  $\alpha_0$  with the **zero frequency**  $\omega_0 := 0$ ,  $\omega_k(L) := [1 - 2(\alpha_k \cos \omega_k - \beta_k \sin \omega_k)L + (\alpha_k^2 + \beta_k^2)L^2]$  corresponds to the **conjugate (harmonic) seasonal frequencies**  $(\omega_k, 2\pi - \omega_k)$ ,  $\omega_k = 2\pi k/S$ , with the associated parameters  $\alpha_k$  and  $\beta_k$ ,  $k = 1, \dots, S^*$ ,  $S^* := \lfloor (S-1)/2 \rfloor$ , and, for  $S$  even,  $\omega_{S/2}(L) := (1 + \alpha_{S/2}L)$  associates the parameter  $\alpha_{S/2}$  with the **Nyquist frequency**  $\omega_{S/2} := \pi$
- The null hypothesis can be partitioned as  $H_0 = \bigcap_{k=0}^{\lfloor S/2 \rfloor} H_{0,k}$ , where

$$H_{0,0} : \alpha_0 = 1, H_{0,S/2} : \alpha_{S/2} = 1, \quad (14)$$

$$H_{0,k} : \alpha_k = 1, \beta_k = 0, k = 1, \dots, S^* \quad (15)$$





# Testing for seasonal non-stationarity under time-varying volatility

- Expanding the composite  $AR(p + S)$  polynomial  $\phi^*(z) := \alpha(z)\phi(z)$  around the zero and seasonal frequency unit roots  $\exp(\pm i2\pi k/S)$ ,  $k = 0, \dots, \lfloor S/2 \rfloor$ , we obtain the auxiliary HEGY regression,

$$\Delta_S x_{S_{n+s}} = \pi_0 x_{0, S_{n+s-1}} + \pi_{S/2} x_{S/2, S_{n+s-1}} + \sum_{k=1}^{S^*} (\pi_{\alpha, k} x_{k, S_{n+s-1}}^\alpha + \pi_{\beta, k} x_{k, S_{n+s-1}}^\beta) + \sum_{j=1}^p \phi_j^* \Delta_S x_{S_{n+s-j}} + \varepsilon_{S_{n+s}}, \quad (16)$$

where the regressors are defined as,  $x_{0, S_{n+s}} := \sum_{j=0}^{S-1} x_{S_{n+s-j}}$ ,

$x_{S/2, S_{n+s}} := \sum_{j=0}^{S-1} \cos[(j+1)\pi] x_{S_{n+s-j}}$ , and

$x_{k, S_{n+s}}^\alpha := \sum_{j=0}^{S-1} \cos[(j+1)\omega_k] x_{S_{n+s-j}}$ , and

$x_{k, S_{n+s}}^\beta := -\sum_{j=0}^{S-1} \sin[(j+1)\omega_k] x_{S_{n+s-j}}$ , in each case for  $k = 1, \dots, S^*$ ,



# Testing for seasonal non-stationarity under time-varying volatility

- Unit roots at the zero, Nyquist and harmonic seasonal frequencies imply that  $\pi_0 = 0$ ,  $\pi_{S/2} = 0$  and  $\pi_{\alpha,k} = \pi_{\beta,k} = 0$ ,  $k = 1, \dots, S^*$ , respectively –  $t$ - and  $F$ - tests.
- Non-stationary volatility in innovations: time deformation aspect to the limiting distributions of the HEGY unit root statistics – incorrect size (even asymptotically) if standard (homoskedastic) critical values are used.
- Solution: standard wild bootstrap or seasonal block wild bootstrap – replicate the correct first-order asymptotic null distributions of each of the HEGY statistics



# Inference on structural breaks in univariate cointegrated models

- Kurozumi and Skrobotov (2018): constructing a **confidence set for the break date** in cointegrating regression by **inverting the test** for the break location.

$$y_t = w'_{b,t}\beta_b + w_{b,t}(\lambda_0)' \delta_b + w'_{f,t}\beta_f + e_t \quad (17)$$

for  $t = 1, \dots, T$ , where  $w_{b,t}$ ,  $w_{b,t}(\lambda_0)$ , and  $w_{f,t}$  are  $p_b$ -,  $p_b$ -, and  $p_f$ -dimensional regressors, respectively,  $w_{b,t}(\lambda_0) = 1(t > [\lambda_0 T])w_{b,t}$  with  $1(\cdot)$  being an indicator function,  $\lambda_0$  is a true break fraction, true break date is  $T_0 = [\lambda_0 T]$

- $w_{b,t}$  and/or  $w_{f,t}$  (fixed regressor) may be  $I(1)$  or  $I(0)$  (allowing DOLS), break in trend
- Test inversion: Constructing a confidence set for the break date by inverting the test for the location of the break point.
- For the unknown break point, we test for

$$H_N : T_0 = T_1 \quad \text{vs.} \quad H_A : T_0 = T_2 \quad (18)$$

with the significance level  $\alpha$ , and if the null hypothesis is accepted, then we



# Inference about structural breaks in univariate cointegrated models

- We cannot directly estimate model by using  $w_{b,t}(\lambda_0)$  because  $w_{b,t}(\lambda_0)$  depends on the unknown break fraction  $\lambda_0$ .
- Maximizing the weighted average of  $P(\varphi \text{ rejects } H_N | \delta, \lambda_2)$  over  $\delta$  and  $\lambda_2$  using some weighting functions, which is given by

$$\int \int P(\varphi \text{ rejects } H_N | \delta, \lambda_2) dQ_{\lambda_2}(\delta) dJ(\lambda_2) \quad (19)$$

where  $Q_{\lambda_2}(\delta)$  and  $J(\lambda_2)$  are non-negative measures on  $\mathbb{R}^{pb}$  and  $(0, 1)$ , respectively.

- Test that maximizes the averaged power given by (19) rejects the null hypothesis when

$$\widetilde{LR}_T(\lambda_1) = \int_{\lambda_2 \in \Lambda_\epsilon} (1+c)^{-p\beta/2} \exp \left\{ \frac{c}{2(1+c)\sigma^2} \right.$$

$$\left. y' M_{w_1} R(\lambda_2, \lambda_1) (R(\lambda_2, \lambda_1)' M_{w_1} R(\lambda_2, \lambda_1))^{-1} R(\lambda_2, \lambda_1) \right\} d\lambda_2 \geq \theta$$



# Inference about structural breaks in univariate cointegrated models

- Limiting distribution of  $\widetilde{LR}_T(\lambda_1)$  for different models
- Dependence on localizing parameter  $c$ .

$$\text{sup-}LR_T(T_1) = \max_{T_2 \in \mathcal{T}_\epsilon} F_{T_2}(T_1), \quad (20)$$

$$\text{avg-}LR_T(T_1) = \frac{1}{T^*} \sum_{T_2 \in \mathcal{T}_\epsilon} F_{T_2}(T_1), \quad (21)$$

$$\text{exp-}LR_T(T_1) = \log \left\{ \frac{1}{T^*} \sum_{T_2 \in \mathcal{T}_\epsilon} \exp \left( \frac{1}{2} F_{T_2}(T_1) \right) \right\}, \quad (22)$$

where  $\mathcal{T}_\epsilon = \{T_2 : \epsilon T \leq T_2 < T_1 - \epsilon T, T_1 + \epsilon T < T_2 \leq (1 - \epsilon)T\}$ ,  $T^*$  is the number of  $T_2$  included in  $\mathcal{T}_\epsilon$ , and  $F_{T_2}(T_1)$  is the test statistic for the simple null hypothesis of  $T_0 = T_1$  against  $T_0 = T_2$ .

# Testing for predictability with possibly non-stationary and endogenous predictors under non-stationary volatility

Ibragimov, Kim and Skrobotov (2023):

- Linear predictive regression model:

$$y_t = \alpha + \beta x_{t-1} + u_t, \quad t = 1, \dots, T, \quad (23)$$

- Volatility model:

$$u_t = v_t \varepsilon_t,$$

- $H_0 : \beta = 0$ : replace  $v_t$  by its consistent estimator  $\hat{\sigma}((t-1)/T)$
- Test statistic:

$$\tau(\hat{\sigma}) = \frac{1}{T^{1/2}} \sum_{t=1}^T \text{sgn}(x_{t-1}) \frac{y_t}{\hat{\sigma}((t-1)/T)}, \quad (24)$$

- Uniform convergence of  $\hat{\sigma}^2(r)$  to  $\sigma_T^2(r)$
- $\text{sign}(x_{t-1})$  – asymptotic normality of  $\tau(\hat{\sigma})$  regardless of endogeneity of  $x_{t-1}$  and nonstationarity



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# Identification the bubbles under non-stationary volatility

Kurozumi, Skrobotov and Tsarev (2020):

- DGP for  $\{y_t\}$  in TVP form:

$$y_t = (1 + \delta_t)y_{t-1} + \varepsilon_t \quad \text{or} \quad \Delta y_t = \delta_t y_{t-1} + \varepsilon_t \quad (25)$$

with the following definition of  $\delta_t$

- 4 regimes:  $\delta_t > 0$  – explosive regime,  $< 0$  – stationary collapsing regime,  $= 0$  – unit root regime.
- Non-stationary volatility:  $\varepsilon_t = \sigma_t z_t$ , where  $\{z_t\}$  is a MDS with respect to natural filtration, and the volatility  $\sigma_t$  is defined as  $\sigma_{\lfloor sT \rfloor} = \omega(s)$  for  $s \in [0, 1]$ , where  $\omega(\cdot) \in \mathcal{D}$  is a non-stochastic and strictly positive function satisfying  $0 < \underline{\omega} < \omega(s) < \bar{\omega} < \infty$ .
- FCLT:

$$\frac{1}{\sqrt{T}} y_{\lfloor rT \rfloor} = \frac{1}{\sqrt{T}} \sum_{t=1}^{\lfloor rT \rfloor} \varepsilon_t \Rightarrow \bar{\omega} W(\eta(r)) =: \bar{\omega} W^\eta(r) \quad (0 \leq r \leq 1), \quad (26)$$

for  $y_{\lfloor \cdot \rfloor}$  defined in (25), where  $\Rightarrow$  denote weak convergence in  $D[0, 1]$  and  $W(\cdot)$  is a standard BM,  $W^\eta(\cdot)$  – variance transformed BM





# Identification and dating the (explosive) bubbles under non-stationary volatility

- Partial sum process of  $\{\varepsilon_t\}$  is asymptotically characterized by the variance profile (Cavaliere and Taylor (2008b)):

$$\eta(s) := \left( \int_0^1 \omega(r)^2 dr \right)^{-1} \int_0^s \omega(r)^2 dr.$$

- Strictly monotonically increasing function  $\Rightarrow$  unique inverse  $g(s) := \eta^{-1}(s)$   
 $\Rightarrow \tilde{y}_t = y_{t'} - y_{t'=0}$  with a non-decreasing sequence  $t' = \lfloor g(t/T)T \rfloor \Rightarrow$

$$T^{-1/2} \tilde{y}_{\lfloor rT \rfloor} \approx T^{-1/2} y_{\lfloor g(\lfloor rT \rfloor / T)T \rfloor} \approx T^{-1/2} y_{\lfloor g(r)T \rfloor} \Rightarrow \bar{\omega} W^\eta(g(r)) = \bar{\omega} W(r) \quad (27)$$

because  $W^\eta(g(r)) = W(\eta(g(r))) = W(r)$



# Identification and dating the (explosive) bubbles under non-stationary volatility

- Time-transformed tests:

$$STADF = \sup_{\tau_2 \in [\tau_0, 1]} TADF_0^{\tau_2} \quad \text{and}$$

$$GSTADF(\tau_0) = \sup_{\tau_2 \in [\tau_0, 1], \tau_1 \in [0, \tau_2 - \tau_0]} TADF_{\tau_1}^{\tau_2},$$

$$\text{where } TADF_{\tau_1}^{\tau_2} = \frac{\tilde{y}_{[\tau_2 T]}^2 - \tilde{y}_{[\tau_1 T]}^2 - \bar{\omega}^2([\tau_2 T] - [\tau_1 T])}{2\bar{\omega} \sqrt{\sum_{t=[\tau_1 T]+1}^{[\tau_2 T]} \tilde{y}_{t-1}^2}}. \quad (28)$$

- The limiting distributions are the same as in the case of homoskedasticity
- $\Rightarrow$  no need any bootstrap procedures to control the size.
- Estimator of  $\eta(s)$ : non-parametrically estimate autoregressive coefficient, collect residuals, and use them for estimating the variance profile.



# Dating the bubble

Kurozumi and Skrobotov (2022):

- Four-regimes bubble model

$$y_t = \mu + u_t, \quad (29)$$

$$u_t = \begin{cases} u_{t-1} + \varepsilon_t, & t = 2, \dots, k_e, \\ (1 + \delta_1)u_{t-1} + \varepsilon_t, & t = k_e + 1, \dots, k_c, \\ (1 - \delta_2)u_{t-1} + \varepsilon_t, & t = k_c + 1, \dots, k_r, \\ u_{t-1} + \varepsilon_t, & t = k_r + 1, \dots, T, \end{cases} \quad (30)$$

- a unit root process until the time  $k_r$ ,
- then explosive process until  $k_c$ ,
- then after  $k_c$ , there may be a stationary collapsing regime (which is interpreted as the return to normal market behavior) until the time  $k_r$ ,
- then again unit root process.



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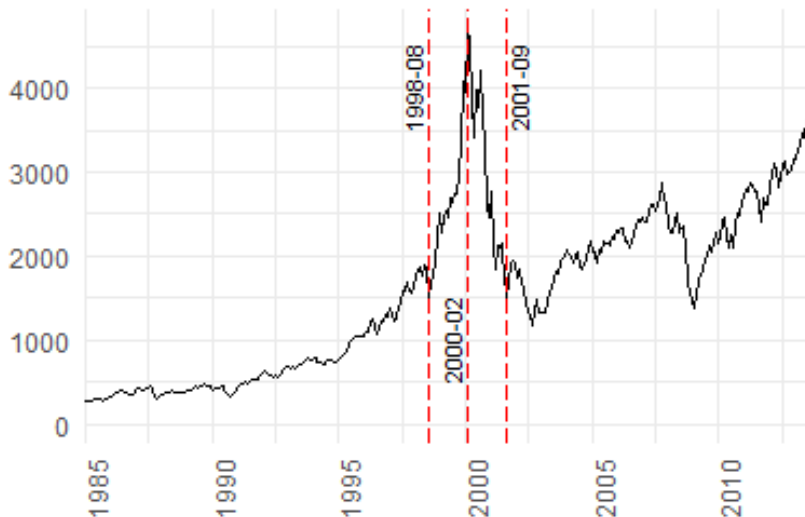
# Dating the bubble

Estimation of the break dates:

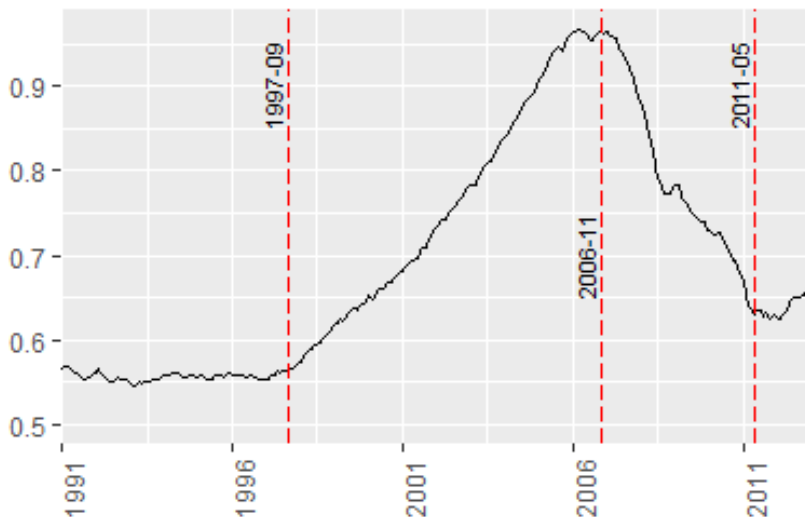
- Harvey et al. (2017): minimisation of SSR – computationally expensive
- Phillips et al. (2015): recursive estimation – inaccurate
- Our **sample splitting** approach: First, estimate the break date for misspecified **two-regime model** minimizing SSR  $\Rightarrow$  the obtained break date is consistent for the date of collapse:  $\lim_{T \rightarrow \infty} P(\hat{k} = k_c) = 1$
- Second, **split the sample into two parts** and estimate  $\tau_e$  and  $\tau_r$
- For the estimation of  $k_r$ , we minimize the sum of the squared residuals using the **second sub-sample**
- Asymptotic property of  $\hat{k}_r$ : **two cases:  $\delta_1 > \delta_2$  (consistency of  $\hat{k}_r$ ) and  $\delta_1 < \delta_2$  (asy distribution of  $\hat{k}_r$ ).**
- **Non-stationary volatility**: robust, but can be extended by using WLS in minimisation



# Empirical application: NASDAQ Composite Index



# Empirical application: U.S. house price index



КОН

ІДАРА



# Reviews

- Skrobotov (2020) (Applied Econometrics) – review on structural breaks in unit root testing
- Skrobotov (2021a,b) (Applied Econometrics) – two reviews on testing and inference in cointegration models (univariate and multivariate)
- Skrobotov (2023) (Dependence Modelling) – review on bubbles





# Empirics

## Skrobotov and Fokin (2018)

- asymmetric reaction of the Bank of Russia to the positive and negative shocks of external economic conditions from 1999 to 2014
- TVECM: nonlinear cointegrating regression with the real exchange rate and real oil prices
- regimes are depended on the sign of the oil price shock

## Polbin and Skrobotov (2021)

- aggregated consumption function for Russia, households consume a constant fraction of a permanent income
- structural break in the parameter of the propensity to consume, endogeneity
- the parameter of the propensity to consume of permanent GDP decreased by 6.5-9.2% after 2014



# Empirics

## Dobronravova, Perevyshin, Skrobotov and Shemyakina (2019)

- 33 food products prices in 78 Russian regions from 2003 to 2018
- unit root null hypothesis versus a nonlinear alternative, and the null of linearity against a nonlinear alternative
- About a 40% of time series corresponds to the three-regime TAR-model and weak form of the law of one price.

## Perevyshin and Skrobotov (2017)

- law of one price in 76 Russian regions for 69 goods
- Panel unit root tests: evidence in favor of the law of one price for most food products, medicines, household chemicals and some of the services provided by public companies



# Contributions

The results of the dissertation research were used in the following projects:

- Russian Science Foundation Project No. 19-18-13029 "Modern methods of robust inference in finance and economics, with applications to the study of crises and their propagation in financial and economic markets" 2017-2020, principal investigator;
- Russian Science Foundation Project No. 20-78-10113 "New methods of robust inference for developing markets: Financial bubbles, time-varying volatility, structural breaks and beyond" 2020-2023, leading investigator.

More than 20 conferences, including: 10-16th International Conference on Computational and Financial Econometrics, 2nd, 4th, 5th, 7th Annual Conference of the International Association for Applied Econometrics, World Congress of Econometric Society 2021, among others, 12th International Vilnius Conference on Probability Theory and Mathematical Statistics and 2018 IMS Annual Meeting on Probability and Statistics.



# List of author's original articles

- Confidence Sets for the Break Date in Cointegrating Regressions (with E. Kurozumi) // Oxford Bulletin of Economics and Statistics, 2018, 80, 514-535. [Q1]
- On Bootstrap Implementation of Likelihood Ratio Test for a Unit Root // Economics Letters, 2018, 171, 154-158. [Q2]
- Wild Bootstrap Seasonal Unit Root Tests for Time Series with Periodic Non-Stationary Volatility (with G. Cavaliere and A.M.R. Taylor) // Econometric Reviews, 2019, 38, 509-532. [Q1]



## List of author's original articles

- On Robust testing for Trend // Economics Letters, 2022, 212, In Press [Q1]
- Time-Transformed Test for the Explosive Bubbles under Non-stationary Volatility (with E. Kurozumi and A. Tsarev) // Accepted in Journal of Financial Econometrics [Q1]
- Likelihood ratio test for change in persistence // Accepted in Communications in Statistics – Theory and Methods [Q3]
- On the asymptotic behaviour of the bubble dates estimators (with E. Kurozumi) Accepted in Journal of Time Series Analysis [Q2]
- Testing for Explosive Bubbles: a Review // Accepted in Dependence Modeling [Q3]



# List of author's original articles

- New robust inference for predictive regression (with R. Ibragimov and J. Kim) // Accepted in Econometric theory [Q1]
- The price convergence of individual goods in the Russian regions (with Yu. Perevyshin) // Journal of the New Economic Association, 2017, 35, 71-102. [Q4]
- Testing Asymmetric Convergence of the Real Exchange Rate to Equilibrium During Ruble Exchange Rate Targeting (with N. Fokin) // Economic Policy, 2018, 13, 132–147. [Q3]



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- Limits of regional food price differences and invisible hand (with E. Dobronravova, Y. Perevyshin and K. Shemyakina) // Applied Econometrics, 2019, 53, 30-54. [Q3]
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