On Bootstrap Implementation of Likelihood Ratio Test for a Unit Root

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Abstract

In this paper we investigate the bootstrap implementation of the likelihood ratio test for a unit root recently proposed by Jansson and Nielsen (2012). We demonstrate that the likelihood ratio test shows poor finite sample properties under strongly autocorrelated errors, i.e. if the autoregressive or moving average roots are close to -1. The size distortions in these case are more pronounced in comparison to the bootstrap M and ADF tests. We found that the bootstrap version of likelihood ratio test (with autoregressive recolouring) demonstrates better performance than bootstrap M tests. Moreover, the bootstrap likelihood ratio test show better finite sample properties in comparison to the bootstrap ADF in some cases.
Key words: likelihood ratio test, unit root test, bootstrap.
JEL: C12, C22.

1 Introduction

The implementation of bootstrap in unit root testing has a long history in econometric time series literature. The theory of bootstrap metodology in unit root context was developed in, *inter alia*, Basawa *et al.* (1991*a*,*b*) and Park (2002, 2003). Chang and Park (2003) considered sieve-based implementation of the residual-based bootstrap for standard ADF type test and Park (2006) considered a bootstrap theory for weakly integrated processes. Looking beyond the ADF type test, Cavaliere and Taylor (2009*a*) investigated the bootstrap implementation of the so-called M type unit rot tests proposed by Stock (1999). They took attention to the wild bootstrap and heteroskedasticity pattern of underlying errors in their model. It should be noted that Palm *et al.* (2008) review different bootstrap unit root tests. Smeekes (2012) analysed the role of detrending in the first step of the bootstrap algorithm and also in the bootstrap recursion. He considered OLS, GLS and recursive detrending procedures.

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Another field of research is the development of optimal tests. The seminal paper is Elliott *et al.* (1996) (hereafter ERS) who developed a nearly (asymptotically) efficient test and obtained Gaussian asymptotic power envelopes for unit root tests. Recently Jansson and Nielsen (2012) developed the likelihood ratio test derived from the full likelihood (in contrast to conditional likelihood, as in standard ADF test, conditioning being with respect to the initial observation). The proposed likelihood ratio test is also nearly (asymptotically) efficient but not asymptotically equivalent to the test proposed by ERS. The likelihood ratio test was further extended to a seasonal unit root context (Jansson and Nielsen (2011)) and cointegration context (Boswijk *et al.* (2015)).

The standard problem in unit root testing is serious size distortion when the autoregressive or moving average roots are close to -1. Serious size distortions of Jansson and Nielsen (2012) test are even more pronounced than conventional ADF and M tests in some cases. We propose to use the sieve-based bootstrap version (recoloured bootstrap) in the spirit of Chang and Park (2003) and Cavaliere and Taylor (2009a). Monte-Carlo simulations show the good finite sample properties of the bootstrapped version of Jansson and Nielsen (2012) which outperforms the bootstrap ADF test in some cases.

The paper is organized as follows. In Section 2 we formulate the model and likelihood ratio test statistics. The bootstrap algorithm and its asymptotic properties are discussed in Section 3. The finite sample properties via Monte-Carlo simulations are presented in Section 4. Section 5 concludes. All proofs and additional Monte-Carlo results are contained in the Supplementary Appendix.

2 Likelihood ratio test

Consider the following data generating process (DGP):

$$y_t = \beta' d_t + u_t,\ t = 1, \ldots, T, \tag{1}$$

$$u_t = \rho u_{t-1} + \varepsilon_t, \tag{2}$$

$$\phi(L)\varepsilon_t = e_t, \tag{3}$$

where $d_t = 1$ (constant case) or $d_t = (1, t)'$ (trend case), β is an unknown parameter. The kth order lag polynomial $\phi(z)$ satisfies: (a) $0 \leq p < \infty$, and (b) $\phi(z) \neq 0$ for all $|z| \leq 1$. This assumption is standard and imposes that $\phi(z)$ is a stationary finite-order polynomial, but this assumption can be weakened without changing our main results. The innovation process e_t is a martingale difference (with some filtration $\mathcal{F}_t$) with $E(e_t^2|\mathcal{F}_{t-1}) = \sigma^2$ and $E|e_t|^r < K < \infty$ for $r \geq 4$. The initial condition is assumed to be $u_0 = o_p(T^{1/2})$.

Our purpose is to test the null hypothesis of a unit root, $H_0 : \rho = 1$ against the stationary alternative $H_1 : |\rho| < 1$. Recently Jansson and Nielsen (2012) proposed nearly efficient likelihood ratio unit root test for the testing this null hypothesis H_0. Let the log likelihood function associated with the model (1)-(3) with $u_1 = \cdots = u_{-p} = 0$ (up to the constant) be[1]

$$L(\rho, \lambda, \beta; \sigma^2, \phi) = -\frac{T}{2}\log\sigma^2 - \frac{1}{2\sigma^2}(Y_{\rho,\phi} - D_{\rho_1,\rho_2,\phi}\beta)'(Y_{\rho,\phi} - D_{\rho,\phi}\beta), \tag{4}$$

where $y_0 = \cdots = y_{-p} = 0$, $d_0 = \cdots = d_{-p} = 0$, and p is the order of the polynomial $\phi(L) = 1 - \phi_1 L - \cdots - \phi_p L^p$. Matrices $Y_{\rho,\phi}$ and $D_{\rho,\phi}$ are defined as $(1-\rho L)\phi(L)y_t$ and $(1-\rho L)\phi(L)d_t'$,

[1] We suppress the dependence of all terms on T for notational convenience.

respectively. Then the likelihood ratio test statistic is written as

$$\begin{aligned} LR &= \max_{\bar\rho\le 1,\beta} L(\bar\rho,\beta;\hat\sigma^2,\hat\phi^2) - \max_{\beta} L(1,1,\beta;\hat\sigma^2,\hat\phi^2) \\ &= \max_{\bar\rho} \mathcal{L}(\bar\rho;\hat\sigma^2,\hat\phi^2) - \mathcal{L}(1;\hat\sigma^2,\hat\phi^2), \end{aligned} \tag{5}$$

where $\hat\sigma^2$ and $\hat\phi$ are the estimators of σ^2 and $\phi = (\phi_1,\ldots,\phi_k)'$, respectively, and

$$\mathcal{L}(\rho;\sigma^2,\phi^2) = -\frac{T}{2}\log\sigma^2 - \frac{1}{2\sigma^2}Y'_{\rho,\phi}Y_{\rho,\phi} + \frac{1}{2\sigma^2}(Y'_{\rho,\phi}D_{\rho,\phi})(D'_{\rho,\phi}D_{\rho,\phi})^{-1}(D'_{\rho,\phi}Y_{\rho,\phi}) \tag{6}$$

is the profile log likelihood function obtained by maximizing $L(\rho,\beta;\sigma^2,\phi^2)$ with respect to the nuisance parameter β, which is related to the deterministic component. Substituting the consistent estimators $\hat\sigma^2$ and $\hat\phi$ into (5), the LR statistic is maximized with respect to the only one parameter, ρ, although there is no closed form expression for this statistic. We discuss the choice of $\hat\sigma^2$ and $\hat\phi$ in Section 4.

Jansson and Nielsen (2012) found the limiting distribution of the LR^i test statistic ($i=\mu$ for the constant case and $i=\tau$ for the trend case) under the null hypothesis. In the constant case ($d_t = 1$), we have

$$LR^\mu \Rightarrow \max_{\bar c\le 0}\Lambda(\bar c) =: \xi^\mu, \tag{7}$$

where

$$\Lambda(\bar c) = \bar c\mathcal{S} - \frac{1}{2}\bar c^2\mathcal{H},\ \mathcal{S} = \frac{1}{2}(W(1)^2-1),\ \mathcal{H} = \int_0^1 W(r)^2 dr.$$

In the trend case ($d_t = (1,t)'$), we have

$$LR^\tau \Rightarrow \max_{\bar c\le 0}\Lambda^\tau(\bar c) =: \xi^\tau, \tag{8}$$

where

$$\Lambda^\tau(\bar c) = \Lambda(\bar c) + \frac{1}{2}\frac{\left((1-\bar c)W(1) + \bar c^2\int_0^1 rW(r)dr\right)^2}{1-\bar c+\bar c^2/3} - \frac{1}{2}W(1)^2.$$

Remark 1 We note that in the constant case the limiting distribution of the LR^μ test coincides with the case of the absence of the deterministic term. This is a standard result for GLS-based tests.

Remark 2 As Jansson and Nielsen (2012) noted in their Discussion section, the LR test can not be interpreted a an (asymptotically) point optimal test of ERS. This is because the LR test can be expressed as point optimal test with $\hat c_{LR}$ as argument, where $\hat c_{LR}$ is based on maximising the likelihood function under the alternative hypothesis. However, this estimate $\hat c_{LR}$ is random in the limit. Also, while the limiting distribution of the LR test coincides with the limiting distribution of the ADF-GLS test proposed by ERS in the constant case, this is not true for the trend case. This is due to the ADF-GLS test using a plug-in estimator of β first based on a fixed non-centrality parameter, say, $\bar c_{ERS}$. Then this test profiles out other nuisance parameters (σ^2 and ϕ). The LR test, conversely, uses plug-in estimators of σ^2 and ϕ, and then profiles out β which affect the limiting distribution. Note that these distinctions of the tests are virtually negligible in terms of local asymptotic power. See Jansson and Nielsen (2012, Section 3) for more discussion.

3 Bootstrap Likelihood Ratio Test

As we will see in the simulation section, the likelihood ratio test suffers from serious size distortions when the errors are strongly autocorrelated. This phenomenon is already observed for many other unit root tests (see, e.g., Cavaliere and Taylor (2009a) for so called M unit root tests). The extensive literature proposed to use bootstrap methodology to correct the inference under the null hypothesis. In order to eliminate the influence from serial correlation, in this section we consider sieve based implementation (recoloring) of the bootstrap likelihood ratio unit root test of Jansson and Nielsen (2012).[2]

In this study, we focus on residual based i.i.d. bootstrap of Chang and Park (2003), Park (2002, 2003) with sieve based recoloring.[3] First we need to define residuals for our botstrap algorithm. Define (quasi) GLS-residuals $\hat{u}_t^i$ $(i = \mu, \tau)$ as

$$\hat{u}_t^i = y_t - \hat{\beta}_{i,GLS}' d_t,$$

where $\hat{\beta}_{i,GLS}'$ is the OLS estimate from the regression of $\boldsymbol{y}^{\bar{\rho}} = [y_1, (1-\bar{\rho}L)\, y_2, \ldots, (1-\bar{\rho}L)\, y_T]'$ on $\boldsymbol{Z}^{\bar{\rho}} = [d_1, (1-\bar{\rho}L)\, d_2, \ldots, (1-\bar{\rho}L)\, d_T]'$ with $\bar{\rho} = 1 - \bar{c}/T$ and $\bar{c} = 7$ for the constant case and $\bar{c} = 13.5$ for the trend case. Next define $\hat{e}_t$ as residuals from the fitted ADF regression

$$\Delta \hat{u}_t^i = \hat{\rho}\hat{u}_{t-1}^i + \sum_{j=1}^{p} \hat{\phi}_j^i \Delta \hat{u}_{t-j}^i + \hat{e}_t^i. \tag{9}$$

Next we outline the bootstrap algorithm.

Algorithm 1 (Bootstrap LR Test)

Step 1: *Obtain the standard LR test statistics, LR^i $(i = \mu, \tau)$ along with the corresponding OLS residuals, $\hat{e}_t^i$, from the (quasi) GLS-detrended ADF regression* (9)*. Set $\hat{e}_t^i = 0$ for $t = 1, \ldots, p+1$.*

Step 2: *Generate the vectors of i.i.d. bootstrap errors $\{e_t^*\}_{t=1}^T$ according to resampling from the centered residuals $(\hat{e}_t^i - \overline{\hat{e}^i})$, where $\overline{\hat{e}^i} = T^{-1}\sum_{t=1}^T \hat{e}_t^i$.*

Step 3: *Construct u_t^* through the recursion*

$$u_t^* = \sum_{j=1}^{p} \hat{\phi}_j u_{t-j}^* + e_t^*,\ t = p+1, T,$$

using estimated parameters $\hat{\phi}_j$ from the regression (9) *initialized at $u_1^* = \cdots = u_p^* = 0$ and then construct the bootstrap sample data through the recursion*

$$\Delta y_t^* = u_t^*,\ t = 2, \ldots, T,$$

initialized at $y_1^* = 0$.

[2] Preliminary simulation showed that the same bootstrap likelihood ratio tests, but without recoloring, do not allow us to fix the size distortions under strongly autocorrelated errors. We do not provide these results to save space.

[3] We also could implement wild bootstrap schemes according Cavaliere and Taylor (2008, 2009a,b) if we suspect a conditionally heteroskedasticity or non-stationary volatility in the errors.

Step 4: *Using the bootstrap sample, $\{y_t^*\}$, compute the bootstrap LR statistics, denoted $LR^{*,i}$ $(i = \mu, \tau)$ exactly as was done for the original data in Step 1 (including any de-trending), for some fixed lag length $p^* \geq 0$.*

Step 5: *Bootstrap p-values are then defined as: $P_{i,T}^* := G_{i,T}^*(LR^i)$ $(i = \mu, \tau)$, where $G_{i,T}^*(\cdot)$ $(i = \mu, \tau)$ denote the conditional (on the original sample data) cumulative distribution functions (cdf's) of $LR^{*,i}$. In practice, the cdf's required here will be unknown, but can be approximated in the usual way through numerical simulation.*

Remark 3 In Step 1, we use the residuals $\hat{e}_t^i$ from the (quasi) GLS-detrended ADF regression (9). Smeekes (2012) investigated the effect of the detrending method on asymptotic validity of the bootstrap algorithms. He demonstrated that the bootstrap test was valid for different methods of detrending (recursive or full-time and OLS or GLS) in the first step of the algorithm for generating bootstrap errors. The detrending matters only for constructing bootstrap test statistics due to asymptotic properties of the original asymptotic tests. Moreover, there are other ways to generate the bootstrap sample. For example, we can impose the unit root null hypothesis to obtain $\hat{e}_t^i$ (see Remark 4 of Cavaliere and Taylor (2009a)). But the simulation results seem to be very similar across different methods and we do not report on them. The full set of the simulation results is available upon request.

Remark 4 In Step 4, the lag length p^* for the bootstrap tests should not be lower than k from the orignal tests. In our simulations, we just set $p^* = p$ as is common in the literature. However, there are studies (see, e.g., Smeekes and Taylor (2012)) where p^* is choosen in each bootstrap replication.

We now proceed to the asymptotic properties of our proposed bootstrap algorithm.

Proposition 1 *Let the assumptions of* (1)-(3) *hold. Then $LR^{*,i} \Rightarrow_p \xi^i$ $(i = \mu, \tau)$, where $\Rightarrow_p$ is used to denote weak convergence in probability in the sense of Giné and Zinn (1990). Also, $P_T^* \xrightarrow{w} U[0,1]$, where P_T^* is again used generically to denote any of the bootstrap p-values of the LR tests, and $U[0,1]$ denotes a uniform distribution on* $[0,1]$.

Remark 5 The Proposition 1 implies that the bootstrap test $LR^{*,i}$ $(i = \mu, \tau)$ attains the same first-order limiting null distribution as the corresponding original test LR^i. Moreover, the bootstrap p-values are (asymptotically) uniformly distributed under the null hypothesis, implying that the test is asymptotically size controlled.

Remark 6 It can be shown that Proposition 1 also hold under local alternative of the form $\rho = 1 - c/T$ for $c \geq 0$ and, therefore, the bootstrap LR test has the same local asymptotic power as the local asymptotic power of the size-adjusted original test. Under the fixed alternative, the bootstrap LR test has the same consistency properties as the original LR test.

4 Finite Sample Size and Power

In this section we investigate the finite sample behavior of our proposed bootstrap likelihood ratio test. We compare this test with commonly used bootstrap unit root tests proposed by Cavaliere and Taylor (2009a) (so-called M-tests)[4] and Chang and Park (2003) (augmented Dickey-Fuller test), both with autoregressive recolouring.

Our simulations are based on the following DGP (we assume no deterministic component without loss of generality):

$$\rho(L)y_t = \varepsilon_t, \tag{10}$$
$$\phi(L)\varepsilon_t = \theta(L)e_t, \tag{11}$$

with $y_1 = \varepsilon_1 = 0$ and $e_t \sim i.i.d.N(0,1)$. The deterministic term is set to be zero without loss of generality. We generate the data for samples of $T = 50, 100$ and 200 with 10,000 Monte-Carlo replications. For analysis of the weak dependence of the errors we focus on (first-order) autoregressive errors ($\phi(L) = 1-\phi L$, $\theta(L) = 1$) and (first-order) moving average errors ($\phi(L) = 1$, $\theta(L) = 1 + \theta L$). The results are reported for $\phi = \{-0.8, -0.5, 0.5, 0.8\}$ in the AR(1) case and $\theta = \{0.8, 0.5, -0.5, -0.8\}$ in the MA(1) case. We also report results for the IID case ($\phi(L) = 1$, $\theta(L) = 1$).

For implementation of the LR tests we use two different popular types of the consistent estimators of σ^2 and ϕ. The first is recommended in (5), $\hat{\sigma}^2 = (T-p-1)^{-1}\sum_{t=k+1}^{T}(\Delta y_t - \hat{\kappa}' Z_t)^2$ and $\hat{\phi} = (0, I_k)\hat{\kappa}$, where $\hat{\kappa}$ is the OLS-estimator in the regression of Δy_t on $Z_t = (\Delta d_t', \Delta y_{t-1}, \ldots, \Delta y_{t-k})'$. These estimators impose the null hypothesis on the regression ADF regression. We denote the LR test with this type of estimators as LR_1^i. The second type uses the ADF-regression under the alternative hypothesis, so that $\hat{\sigma}^2 = (T-k-1)^{-1}\sum_{t=k+1}^{T}(\hat{e}_t^i)^2$ and $\hat{\phi} = (\hat{\phi}_1^i, \hat{\phi}_1^i, \ldots, \hat{\phi}_k^i)'$ from the regression (9). We denote the LR test with this type of estimators as LR_2^i.

We compare standard LR tests (LR_1 and LR_2) with their bootstrap counterparts (LR_1^* and LR_2^*) and classical bootstrap tests, MZ_ρ^*, MSB^*, MZ_t^*, MP_t^* and ADF^*. Table 1 represents the size of these tests in the constant case (here $\rho(L) = 1 - L$). For the IID case the size of all tests is close to the nominal one (although slightly undersized). The size of LR_1^* and LR_2^* is closer to the nominal one than the other bootstrap tests. For negative AR(1) errors, LR_2 is oversized but this oversizing vanishes for its bootstrap counterpart. Again, the LR_1^* and LR_2^* control size well while we find serious undersizing for so-called M-tests. For positive AR(1) errors the both LR_1 and LR_2 are seriously undersized especially for $\phi = 0.8$ even for $T = 100$ and 200. This problem is fixed by applying bootstrap. The size of LR_1^* and LR_2^* is closer to the nominal one than all other tests.

Next consider MA(1) errors. For positive θ we observe greatly improved size for bootstrap LR tests in comparison to their non-bootstrap counterparts. Again, the size is better than other bootstrap tests considered. For negative θ we have the standard problem of the oversizing of non-bootstrap tests, especially for LR_2. The size of LR_1^* and LR_2^* is considerably smaller than LR_1 and LR_2, respectively. However, it is still larger when compared to all other bootstrap tests. Overall, the size of LR_1^* and LR_2^* is better for all cases except for a negative MA term. Also, the size improves if the sample size increases for all cases.

Table 2 represents the finite sample power of all tests. We investigate the power of the tests under a near integrated alternative, so that $\rho(L) = 1 - (1 - c/T)L))$ with $c = 7$ as is common in unit root literature. For AR(1) errors the power of LR_1^* and LR_2^* is the best for all cases except the case of $\phi = -0.8$, where the ADF^* test outperforms all other tests. The power of M-tests is considerably lower than LR_1^*, LR_2^* and ADF^*. For MA(1) errors, LR_1^* and LR_2^* outperform all

[4] We note that while Cavaliere and Taylor (2009a) investigate the wild bootstrap with heteroskedastic errors, their results on stationary autoregressive errors dynamic also hold.

other tests when the MA parameter θ is positive. For negative θ, the ADF^* test has greater power. Again, all M-tests have lower power than LR_1^*, LR_2^* and ADF^*. Comparing two bootstrap LR tests, LR_1^* and LR_2^*, we observe very similar power properties except for the case of negative θ. In this case LR_2^* outperforms LR_1^*. Refering to Table 1, we observe, that in this case LR_2^* has a size closer to the nominal one in comparison to LR_1^*.

Tables S.1 and S.2 in Supplementary Appendix show finite sample size and power, respectively, for the trend case (with $c = 13.5$ in $\rho(L) = 1 - (1 - c/T)L)$). Quantitatively, the results are similar to the constant case, but we have a greater performance of the LR_2^* test in comparison to LR_1^* and ADF^*.

Summarizing results, we found that the LR_2^* test has great finite sample properties and can be performed in conjunction with the popular bootstrap ADF test. The latter test outperforms the former in the cases of negative MA and very negative AR components across all cases considered.

5 Conclusion

In this paper we investigated the bootstrap implementation of the likelihood ratio test for a unit root with different types of nuisance parameter estimators. It was shown that while the standard LR test shows serious size distortion, its bootstrap counterpart has good finite sample properties. We concluded that the bootstrap LR test can be used as a complement for the bootstrap ADF test with GLS detrending.

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Table 1. Finite sample size (constant case) $-\ c = 0$

	ϕ/θ	T	LR_1	LR_2	LR_1^*	LR_2^*	MZ_ρ^*	MZ_t^*	MSB^*	MP_t^*	ADF^*
IID	0	50	4.0	5.5	3.6	3.6	3.3	3.3	3.4	3.3	3.5
		100	3.4	4.3	3.7	3.6	3.4	3.3	3.3	3.4	3.5
		200	3.9	4.4	4.2	4.2	4.0	4.1	4.1	4.1	4.2
AR(1)	-0.8	50	4.1	10.2	3.3	3.6	0.5	0.4	0.6	0.6	2.9
		100	3.6	6.2	3.1	3.0	1.5	1.4	1.7	1.7	2.9
		200	4.3	5.7	3.9	3.9	3.4	3.3	3.5	3.5	4.0
	-0.5	50	4.0	8.8	3.7	3.8	2.1	2.2	2.2	2.2	3.2
		100	4.1	6.4	3.9	3.9	3.3	3.2	3.3	3.3	3.7
		200	4.1	5.2	3.9	3.9	3.5	3.5	3.6	3.7	3.7
	0.5	50	0.3	2.0	3.2	3.1	1.6	1.6	1.7	1.7	2.1
		100	1.3	2.6	3.6	3.6	2.8	2.8	2.9	3.0	3.2
		200	2.4	3.3	3.8	3.9	3.5	3.5	3.6	3.7	3.9
	0.8	50	0.1	0.6	4.1	4.2	1.0	1.0	1.0	0.9	3.0
		100	0.3	1.0	3.8	3.7	2.8	2.8	2.6	2.6	3.3
		200	1.6	2.5	4.4	4.4	4.0	4.1	4.0	4.0	4.2
MA(1)	0.8	50	0.5	3.0	3.3	3.0	0.7	0.7	0.7	0.7	1.4
		100	0.7	3.2	3.6	3.3	1.2	1.3	1.3	1.3	2.2
		200	1.4	3.9	3.9	3.6	2.3	2.4	2.4	2.5	3.5
	0.5	50	0.7	2.8	2.9	2.5	1.2	1.2	1.2	1.3	1.7
		100	1.3	3.3	3.3	3.2	2.5	2.5	2.5	2.5	2.9
		200	2.6	4.0	4.1	3.9	3.5	3.7	3.5	3.4	3.7
	-0.5	50	8.5	17.8	8.9	8.6	5.6	5.7	5.6	5.5	7.4
		100	6.3	11.8	6.7	6.5	4.4	4.1	4.5	4.6	5.8
		200	5.8	8.6	5.5	5.4	4.4	4.4	4.5	4.5	5.1
	-0.8	50	26.0	47.0	28.2	27.2	16.6	16.3	16.5	16.6	23.0
		100	13.2	27.8	14.3	12.8	5.6	5.4	5.7	5.7	11.1
		200	10.5	19.9	9.9	8.4	3.4	3.4	3.4	3.5	8.2

Table 2. Finite sample power (constant case) $-$ $c = 7$

	ϕ/θ	T	LR_1	LR_2	LR_1^*	LR_2^*	MZ_ρ^*	MZ_t^*	MSB^*	MP_t^*	ADF^*
IID	0	50	35.0	46.0	32.6	32.4	29.3	26.0	30.3	30.4	30.8
		100	35.7	42.7	36.9	36.7	35.0	32.4	35.5	35.7	36.5
		200	39.4	43.3	40.8	40.7	39.3	37.4	40.0	40.1	40.5
AR(1)	-0.8	50	26.3	57.8	22.6	28.2	4.2	3.3	4.5	4.8	25.8
		100	30.6	48.2	27.6	28.8	17.7	15.3	18.7	19.0	32.4
		200	37.8	46.7	35.3	35.4	33.1	30.7	33.9	33.9	39.1
	-0.5	50	29.3	56.6	27.4	28.2	17.7	15.5	18.5	19.1	26.8
		100	33.1	48.4	31.4	31.7	28.9	26.9	29.5	29.6	33.2
		200	39.8	48.6	38.4	38.2	37.8	35.5	38.5	38.5	40.0
	0.5	50	4.5	16.0	18.1	16.7	8.0	7.0	8.6	9.7	11.8
		100	17.3	29.5	33.2	32.7	27.5	25.2	28.4	28.7	30.1
		200	28.3	36.2	39.0	38.8	35.6	33.6	36.7	36.8	37.8
	0.8	50	0.8	7.7	20.8	20.1	6.8	6.1	7.3	8.5	14.7
		100	4.9	12.4	27.1	26.6	18.2	16.6	19.1	20.1	23.6
		200	16.4	24.1	34.6	34.4	29.3	27.2	30.3	30.9	32.8
MA(1)	0.8	50	5.3	24.7	18.6	17.3	4.9	4.4	5.2	5.9	10.6
		100	8.6	29.2	26.4	25.0	13.0	12.0	13.5	14.2	20.5
		200	17.1	36.8	33.4	32.1	23.3	21.8	24.1	24.7	30.2
	0.5	50	9.1	24.3	17.7	16.5	8.3	7.3	9.0	9.8	11.9
		100	16.9	33.6	30.5	29.5	22.9	21.2	23.6	24.3	27.3
		200	25.6	36.9	35.4	34.6	30.4	28.5	31.4	31.9	33.9
	-0.5	50	38.7	72.5	41.4	43.4	28.4	27.0	28.8	29.0	40.2
		100	36.7	62.1	38.8	39.1	30.3	28.9	30.7	30.7	40.1
		200	43.9	59.6	43.4	43.2	39.1	37.1	39.8	39.7	45.0
	-0.8	50	57.0	91.7	61.7	69.7	45.9	45.4	46.1	46.3	64.3
		100	40.6	83.7	45.1	52.8	26.9	26.1	27.1	27.2	55.4
		200	40.9	75.3	40.9	46.9	23.9	22.5	24.6	24.6	54.6

Supplementary Online Appendix

to

On Bootstrap Implementation of
Likelihood Ratio Test for a Unit Root

S.1 Contents

Section S.1 of this supplement contains mathematical proof of Propositions 1. Section S.2 contains additional Monte Carlo results relating to the trend case.

S.2 Proof of Proposition 1

As we base our bootstrap sample exactly in the same way as in Smeekes (2012), we have invariance principle for the bootstrap errors e_t^*:

$$T^{-1/2}\sum_{t=1}^{\lfloor Tr\rfloor} e_t^* \Rightarrow_p \sigma W(r), \tag{S.1}$$

and the bootstrap sample y_t^*,

$$T^{-1/2}y^*_{\lfloor Tr\rfloor} \Rightarrow_p \sigma\psi(1)W(r), \tag{S.2}$$

where $\psi(L) = 1 + \sum_{j=1}^{\infty}\psi_j z^j$ is the inverse of $\phi(L)$ (see proof of Theorem 2 of Smeekes (2012), p. 888). Note that Smeekes (2012) considered a weaker assumption about errors (assuming infinitely increasing lag order).

Therefore, as in the proof of Jansson and Nielsen (2012), let $\hat{d}_t^* = \hat{\phi}^*(1)^{-1}diag(1, 1/\sqrt{T})\hat{\phi}^*(L)d_t$, $\hat{y}_t^* = \hat{\sigma}^{-1*}\hat{\phi}^*(L)y_t^*$. Then, the test bootstrap test statistic $LR^* = \max_{\bar{c}\leq 0} F(\bar{c}, \hat{X}^*)$, where $\hat{X}^* = (\hat{S}^*, \hat{H}^*, \hat{A}^*, \hat{B}^*)$,

$$\hat{S}^* = \hat{\sigma}^{-1*}T^{-1}\sum_{t=2}^{T}\hat{y}^*_{t-1}\Delta\hat{y}^*_t \Rightarrow_p \int_0^1 W(r)dW(r) =: \mathcal{S},$$

$$\hat{H}^* = \hat{\sigma}^{-2*}T^{-2}\sum_{t=2}^{T}\hat{y}^{2*}_{t-1} \Rightarrow_p \int_0^1 W(r)^2 dr =: \mathcal{H},$$

$$\hat{A}^* = (\hat{A}^*(0), \hat{A}^*(1), \hat{A}^*(2)), \quad \hat{B}^* = (\hat{B}^*(0), \hat{B}^*(1), \hat{B}^*(2)),$$

$$\begin{aligned}
\hat{A}^*(0) &= \sum_{t=1}^{T}\Delta\hat{d}_t\Delta\hat{y}^*_t \Rightarrow_p \begin{pmatrix}\mathcal{Y}\\ W(1)\end{pmatrix} =: \mathcal{A}(0),\\
\hat{A}^*(1) &= \sum_{t=1}^{T}(\Delta\hat{d}_t\hat{y}^*_{t-1} + \hat{d}_{t-1}\Delta\hat{y}^*_t) \Rightarrow_p \begin{pmatrix}0\\ W(1)\end{pmatrix} =: \mathcal{A}(1),\\
\hat{A}^*(2) &= \sum_{t=1}^{T}\hat{d}_{t-1}\hat{y}^*_{t-1} \Rightarrow_p \begin{pmatrix}0\\ \int_0^1 rW(r)dr\end{pmatrix} =: \mathcal{A}(2),\\
\hat{B}^*(0) &= \sum_{t=1}^{T}\Delta\hat{d}_t\Delta\hat{d}'^*_t \Rightarrow_p \begin{pmatrix}K & 0\\ 0 & 1\end{pmatrix} =: \mathcal{B}(0),\\
\hat{B}^*(1) &= \sum_{t=1}^{T}(\Delta\hat{d}_t\hat{d}'^*_{t-1} + \hat{d}_{t-1}\Delta\hat{d}'^*_t) \Rightarrow_p \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} =: \mathcal{B}(1),\\
\hat{B}^*(2) &= \sum_{t=1}^{T}\hat{d}_{t-1}\hat{d}'^*_{t-1} \Rightarrow_p \begin{pmatrix}0 & 0\\ 0 & 1/3\end{pmatrix} =: \mathcal{B}(2),
\end{aligned}$$

where limiting results follow from (S.1) and (S.2). As a result, we have $\hat{X}^* \Rightarrow_p \mathcal{X} = (\mathcal{S}, \mathcal{H}, \mathcal{A}, \mathcal{B})$.
The function $F(\bar{c}, x)$, $x = (s, h, a, b)$ can be written as

$$F(\bar{c}, x) = \bar{c}s - \frac{1}{2}\bar{c}h + \frac{1}{2}N(\bar{c}, a)'D(\bar{c}, b)^{-1}N(\bar{c}, a) - \frac{1}{2}N(0, a)'D(0, b)^{-1}N(0, a),$$

$$N(\bar{c}, a) = a(0) - \bar{c}a(1) + \bar{c}^2a(2), \quad D(\bar{c}, b) = b(0) - \bar{c}b(1) + \bar{c}^2b(2),$$

therefore $F(\bar{c}, \hat{X}^*) \Rightarrow_p F(\bar{c}, \mathcal{X}) = \Lambda^\tau(\bar{c})$ for every $\bar{c} \leq 0$. The convergence result $LR^{*,\tau} = \max_{\bar{c}\leq 0} F(\bar{c}, \hat{X}^*) \Rightarrow_p \max_{\bar{c}\leq 0} F(\bar{c}, \mathcal{X})$ follows from the same lines as in the proof in Jansson and Nielsen (2012).

S.3 Additional Monte Carlo Results

This section contains additional Monte Carlo results relating to the trend case. Tables S.1-S.2 give complementary results to those given in Tables 1-2 respectively. The Monte Carlo DGP and set-up of these experiments were otherwise exactly as detailed in Section 4.

Table S.1. Finite sample size (trend case) $-\ c = 0$

	ϕ/θ	T	LR_1	LR_2	LR_1^*	LR_2^*	MZ_ρ^*	MZ_t^*	MSB^*	MP_t^*	ADF^*
IID	0	50	2.8	6.8	3.3	3.3	3.4	3.6	3.3	3.4	3.1
		100	3.0	5.1	3.3	3.3	3.2	3.3	3.2	3.2	3.2
		200	3.2	4.2	3.4	3.3	3.4	3.4	3.4	3.4	3.4
AR(1)	-0.8	50	2.8	21.6	3.8	6.0	0.1	0.1	0.2	0.2	2.5
		100	2.7	10.2	2.9	3.4	0.4	0.4	0.4	0.4	2.6
		200	3.2	7.0	3.2	3.2	1.6	1.4	1.6	1.7	3.1
	-0.5	50	3.1	17.2	4.9	5.5	1.8	1.7	1.8	1.8	3.7
		100	3.1	9.5	3.6	3.6	2.2	2.2	2.2	2.2	3.2
		200	3.2	6.0	3.3	3.3	3.1	3.1	3.0	3.0	3.3
	0.5	50	0.0	0.6	1.6	1.3	0.6	0.7	0.5	0.4	0.7
		100	0.4	3.0	3.3	3.4	2.8	2.9	2.7	2.6	3.0
		200	1.4	3.2	3.4	3.4	3.2	3.2	3.1	3.2	3.2
	0.8	50	0.0	0.2	1.9	1.7	1.0	1.2	0.9	0.7	2.7
		100	0.0	0.4	3.0	3.0	2.6	2.7	2.4	2.4	3.3
		200	0.3	1.1	3.7	3.7	3.5	3.5	3.5	3.4	3.5
MA(1)	0.8	50	0.1	2.5	2.0	2.1	0.3	0.3	0.3	0.2	0.5
		100	0.2	3.8	2.7	2.9	0.6	0.6	0.6	0.7	1.1
		200	0.3	4.5	3.1	2.8	1.1	1.1	1.1	1.1	1.6
	0.5	50	0.2	1.9	1.2	1.2	0.4	0.5	0.3	0.3	0.4
		100	0.4	3.9	2.8	2.8	1.6	1.7	1.7	1.7	2.1
		200	1.1	3.8	2.7	2.6	2.2	2.2	2.2	2.2	2.3
	-0.5	50	11.2	35.7	15.3	16.5	9.5	9.5	9.6	9.5	12.5
		100	5.9	20.9	8.2	8.2	4.5	4.5	4.5	4.5	6.2
		200	4.3	12.4	5.5	5.2	3.7	3.6	3.7	3.8	4.9
	-0.8	50	36.1	76.4	44.7	48.2	28.8	28.7	28.9	28.9	37.6
		100	19.0	55.7	24.9	25.5	11.5	11.4	11.5	11.6	19.4
		200	9.1	35.0	12.8	11.5	3.0	2.9	3.1	3.1	9.0

Table S.2. Finite sample power (trend case) $-$ $c = 13.5$

	ϕ/θ	T	LR_1	LR_2	LR_1^*	LR_2^*	MZ_ρ^*	MZ_t^*	MSB^*	MP_t^*	ADF^*
IID	0	50	30.8	56.8	35.1	35.4	33.6	33.1	33.6	33.7	32.7
		100	33.6	48.5	36.2	36.2	34.8	34.2	35.2	35.3	35.1
		200	36.3	46.2	38.3	38.2	36.9	36.5	37.3	37.5	37.6
AR(1)	-0.8	50	22.3	82.8	29.8	45.2	0.8	0.7	0.8	0.9	24.9
		100	26.6	64.9	29.0	33.9	4.7	4.4	4.9	4.9	29.4
		200	32.5	54.1	33.0	34.0	19.6	19.0	20.1	20.2	34.1
	-0.5	50	23.7	76.1	34.3	39.7	11.9	11.6	12.3	12.4	28.1
		100	27.0	60.1	31.0	32.1	22.1	21.3	22.7	22.9	30.1
		200	33.9	53.4	36.0	36.1	32.4	31.8	32.7	32.8	35.5
	0.5	50	0.6	4.1	2.8	2.5	1.1	1.2	1.2	1.2	1.4
		100	4.2	23.2	21.9	21.5	17.0	16.8	17.0	17.0	18.2
		200	18.8	34.6	33.4	33.2	30.3	29.8	30.6	31.0	31.9
	0.8	50	0.0	1.8	7.0	6.5	2.1	2.1	2.0	1.8	4.6
		100	0.4	5.8	16.2	16.4	11.6	11.7	11.7	11.7	14.2
		200	3.9	12.3	23.2	23.3	20.2	19.8	20.4	20.8	22.1
MA(1)	0.8	50	1.3	18.2	6.6	7.4	1.8	1.6	1.8	1.9	2.2
		100	3.0	30.1	16.1	17.5	6.7	6.5	6.8	7.0	10.3
		200	6.3	37.2	22.8	23.1	12.8	12.6	13.0	13.1	17.5
	0.5	50	3.8	15.5	6.1	6.0	4.5	4.2	4.7	4.7	4.2
		100	6.4	32.6	20.2	20.1	13.3	13.0	13.6	13.5	15.8
		200	15.5	37.2	27.7	27.2	23.5	23.1	23.8	23.9	25.5
	-0.5	50	43.2	89.9	54.9	61.6	37.1	36.8	37.1	37.3	48.5
		100	31.9	77.2	42.4	44.2	26.4	25.8	26.4	26.5	38.3
		200	33.4	67.9	40.4	40.1	29.9	29.3	30.2	30.4	38.9
	-0.8	50	75.2	98.9	82.1	89.9	68.2	68.0	68.2	68.2	78.9
		100	53.8	97.0	63.0	73.4	43.5	43.4	43.5	43.4	62.4
		200	37.7	91.5	49.0	55.2	21.6	21.2	21.9	21.9	49.8