A simple modification of the Busetti-Harvey stationarity tests with structural breaks at unknown time*

Anton Skrobotov†
The Russian Presidential Academy of National Economy and Public Administration

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Abstract

In this paper a modification of the Busetti and Harvey (2001) test with structural break at unknown time is proposed. As the stationarity test with a super-consistent break date estimator is effective under large breaks and the infimum-test is effective under small breaks, although it has serious size distortions under large breaks, we propose a simple decision rule based on pre-testing for the presence of a break. The proposed modification shows good size properties. Also, an extension for the case of multiple structural breaks is proposed.
Key words: KPSS test, infimum test, size distortion, power, pre-testing, structural breaks.
JEL: C12, C22

1 Introduction

The hypothesis testing for stationarity of a series is often used as the opposite of unit root testing for confirmatory analysis. The most common test in this case is the Kwiatkowski *et al.* (1992) test (hereafter KPSS). Extensions for the case of an allowance of structural breaks was considered, *inter alia*, in Busetti and Harvey (2001) and Busetti and Harvey (2003). Busetti and Harvey (2003) analyzed the behavior of the tests if the break date was unknown. The authors considered the two types of tests. The first uses the superconsistent break date estimate as the true break date (the so called two-step test)[1]. The second is constructed as the infimum of the sequence of stationarity statistics for each possible break date with the assumption that the magnitude of the trend break converges to zero at a faster rate than $T^{-3/2}$ (for a break in the level the rate of convergence should be a faster than $T^{-1/2}$), because without this assumption, the limiting distribution of a test statistic will depend not only on the break fraction, but also on its magnitude. Thus the infimum-test is effective under the absence of a break and its power is higher than the test using

*Author thank Fabio Busetti for providing his Ox code.
†Address correspondence to: Institute for Applied Economic Studies, Russian Presidential Academy of National Economy and Public Administration, 82, Vernadsky pr., 117571, Moscow, Russia.
E-mail: antonskrobotov@gmail.com
[1]For superconsistency of the break date we imply the superconsistency of the corresponding break fraction.

the superconsistent break date estimator. However, under a large break, the infimum test will be seriously oversized. In this case, the two-step test with the break date estimate as the true break date will be effective.

Based on the foregoing, the reasonable strategy is to use the infimum-test if the break is a small or zero, and to use the two-step test if the break clearly occurs in the data. Our approach is similar to the procedures proposed in Harvey *et al.* (2012) (hereafter HLT). For the break detection, we propose performing the pre-tests for the significance of the trend break, t_{PY}, proposed by Perron and Yabu (2009), and t_{HLT}, proposed by Harvey *et al.* (2009), as well as the tests for the significance of the level break, proposed by Harvey *et al.* (2010).[2] If the break is insignificant, the infimum-test should be used. If the break is significant, i.e. we obtain clear evidence of the presence of the break, the two-step test should be used. We also consider the possibility of multiple breaks.

This paper is organized as follows. In Section 2, the data generating process (DGP) and test statistics are described, the decision rule based on pre-testing is proposed, and Monte-Carlo simulation results are discussed. The model with possible multiple breaks is investigated in Section 3. Section 4 is the conclusion.

2 Model

Consider the DGP as an unobserved component representation:

$$\begin{aligned} y_t &= d_t'\gamma + u_t + v_t,\ t = 1, \dots, T, && (1) \\ v_t &= v_{t-1} + \eta_t, && (2) \end{aligned}$$

where d_t is some deterministic function and u_t is a stationary process, $\eta_t \sim i.i.d.(0, q)$, where $q = \sigma_\eta^2$ is the signal-to-noise ratio. Then the stationarity null hypothesis is written as $H_0 : q = 0$.

As in Perron (1989) we consider the three types of models: Model 0 (a change in level, or "crash model") for both a non-trending and trending series, Model I (a change in slope), and Model II (a change in both level and slope, mixed effect). Therefore, the deterministic component d_t can be written as:

$$d_t' = \begin{cases} (1, DU_t), & \text{for Model 0} \\ (1, t, DU_t), & \text{for Model 0t} \\ (1, t, DT_t), & \text{for Model I} \\ (1, t, DU_t, DT_t), & \text{for Model II} \end{cases},$$

where $DU_t = \mathbb{I}(t \geq T_b)$, $DT_t = (t - T_b)\mathbb{I}(t \geq T_b)$, $\mathbb{I}(\cdot)$ is the indicator function, and T_b is the break date. The break fraction is defined as $\lambda = T_b/T$. It is supposed that the true break fraction, λ_0, is unknown, but belongs the range of $\Lambda = [\lambda_L, \lambda_U]$, $0 < \lambda_L < \lambda_U < 1$, where λ_L and λ_U are

[2] We do not provide exact formulas for these tests in order to save space. The reader can refer to the cited papers for more information.

trimming parameters. The parameter vector γ is written as

$$\gamma' = \begin{cases} (\mu_0, \mu_1), & \text{for Model 0} \\ (\mu_0, \beta_0, \mu_1), & \text{for Model 0t} \\ (\mu_0, \beta_0, \beta_1), & \text{for Model I} \\ (\mu_0, \beta_0, \mu_1, \beta_1), & \text{for Model II} \end{cases}.$$

We utilize the KPSS statistic for stationarity testing as:

$$S(\lambda) = \frac{T^{-2} \sum_{t=1}^{T} \left(\sum_{s=1}^{t} \hat{u}_s \right)^2}{\hat{\omega}_u^2}, \tag{3}$$

where $\hat{u}_t = y_t - d_t'\hat{\beta}$ are the OLS residuals from the regression of y_t on d_t depending on the type of deterministic component, and $\hat{\omega}_u^2$ is the long-run variance estimator.[3] The limiting distribution of this statistic, depending on the type of deterministic component, is obtained in Busetti and Harvey (2001) (with corrections from Harvey and Mills (2003)).

The first test which we use if the break date is unknown is the two-step test $S(\hat{\lambda})$, where $\hat{\lambda}$ is obtained by minimizing the sum of squared residuals, $\hat{u}_t$, over all possible break dates. This estimator is superconsistent under the stationarity null hypothesis.

The second test, the infimum-test, is constructed by minimizing the sequence of test statistics over all possible break dates. More precisely, this statistic is constructed as

$$MS = \inf_{\lambda \in \Lambda} S(\lambda). \tag{4}$$

Note that for this test the additional assumption is needed that the magnitude of the break converges to zero at a faster rate than the usual Pitman rate. Without this assumption, the limiting distribution of the MS statistic will depend on both the break date and break magnitude, so that under large breaks serious size distortions will occur.

Thus, as we noted in the introduction section, the MS test is effective under small breaks while the $S(\hat{\lambda})$ test is effective under large breaks. The adaptive test is written as:

$$A(B) : \text{Reject } H_0 \text{ if } \begin{cases} MS > cv_{MS} & B < cv_B \\ S(\hat{\lambda}) > cv_{\hat{\lambda}} & B \geq cv_B \end{cases} \tag{5}$$

where B is a pre-test for the break, cv_B is the corresponding critical value, cv_{MS} is the critical value for the MS test, and $cv_{\hat{\lambda}}$ is the critical value for the $S(\hat{\lambda})$ test depending on the $\hat{\lambda}$. For the B test, either Perron and Yabu (2009) or Harvey *et al.* (2009) can be used for the break in trend or Harvey *et al.* (2010) for the break in level.

Also, we investigated the behavior of the test statistics with the additional inclusion of the tests with trend (without a break, as in HLT). Then, when using the pre-test from Harvey *et al.* (2009), a size distortion will occur if the break is too small to be reliably detected and simultaneously too large to seriously distort the size in the test without a break. The situation with the pre-test from

[3] For long-run variance estimation we use the quadratic spectral window and AR(1) prewhitening, see Andrews (1991).

Perron and Yabu (2009) is more disagreeable. Unlike the Harvey *et al.* (2009) test, the Perron and Yabu (2009) test has the correct size under the stationarity null hypothesis, but this size become seriously oversized at a deviation from the null[4]. In other words, the hypothesis for the absence of a break will often be rejected under the alternative, $H_1 : q > 0$, so that in the tests that we considered, the power will be close to the power of $S(\hat{\lambda})$, and the power gain for $\lambda = 0$ will be negligible. Thus, the results with these strategies are not provided for the sake of brevity, and will be made available upon request.

To illustrate the finite sample behavior, consider the following model with a trend break. Let DGP be (1)-(2) with $\beta_1 \in \{0, 0, .01, 0.02, 0.04, 0.1, 0.2, 0.4, 1, 2\}$, $\lambda \in \{0.3, 0.5, 0.7\}$, $u_t \sim i.i.d.N(0, 1)$, and $\eta_t \sim i.i.d.N(0, q)$ with $q = \{0, 0.2^2\}$. The significance level is 0.05, the sample size is $T = 150$, and the number of replications is 10,000. The results are given in Table 2. Throughout, the $A(t_{HLT})$ does not exceed the $A(t_{PY})$ in terms of power, although the size is somewhat oversized under very small breaks in the cases of $\lambda = 0.3$ and $\lambda = 0.7$. At the same time, the MS has a serious size distortion under large breaks, as in Busetti and Harvey (2003), and $S(\hat{\lambda})$ has the correct size only under large breaks. The $A(t_{HLT})$ test inherits the high power of the MS under small breaks and maintains the correct size under large breaks.

3 Possible multiple breaks

In this section we consider the possibility of more than one structural break. The two break case is investigated in Busetti and Harvey (2001) and Carrion-i-Silvestre and Sansó-i-Rosselló (2005). It is obvious that if there are more breaks than are taken into account when constructing tests, then the size increases to unity as the break magnitudes increases. In case of m breaks, the deterministic component, $d_t'\gamma$, can be written as

$$d_t'\gamma = \mu_0 + \beta_0 t + \boldsymbol{\mu}'\boldsymbol{DU}_t(\boldsymbol{\lambda_0}) + \boldsymbol{\beta}'\boldsymbol{DT}_t(\boldsymbol{\lambda_0}), \tag{6}$$

where $\boldsymbol{DU}_t(\boldsymbol{\lambda_0}) = [DU_t(\lambda_{0,1}), \ldots, DU_t(\lambda_{0,m})]'$ and $\boldsymbol{DT}_t(\boldsymbol{\lambda_0}) = [DT_t(\lambda_{0,1}), \ldots, DT_t(\lambda_{0,m})]'$, and the elements of this vector for the generic break fraction $\lambda_{0,i}$ are expressed as $DU_t(\lambda_{0,i}) = \mathbb{I}(t > \lfloor\lambda_{0,i}T\rfloor)$ and $DT_t(\lambda_{0,i}) = (t - \lfloor\lambda_{0,i}T\rfloor)\mathbb{I}(t > \lfloor\lambda_{0,i}T\rfloor)$ respectively, $\boldsymbol{\mu} = [\mu_1, \ldots, \mu_m]'$ and $\boldsymbol{\beta} = [\beta_1, \ldots, \beta_m]'$ are the parameter vectors. It is assumed that the break fraction $\lambda_{0,i} \in \Lambda = [\lambda_L, \lambda_U]$, $0 < \lambda_L < \lambda_U < 1$, and also that $|\lambda_{0,i} - \lambda_{0,j}| > \varepsilon > 0$ for all $i \neq j$. Moreover, $m \leq 1 + \lfloor(\lambda_U - \lambda_L)/\varepsilon\rfloor$.

If the dating of the breaks is known, then the KPSS statistic $S^m(\boldsymbol{\lambda})$ is constructed as in (3). If the dating of the breaks is unknown, then the situation is similar to the case of one break, and two test statistics can be constructed: $S^m(\hat{\boldsymbol{\lambda}})$, where $\hat{\boldsymbol{\lambda}}$ is the estimated vector of break fractions obtained by minimizing the sum of squared residuals over all possible break dates, and MS^m, which is constructed as the infimum of the sequence of the test statistics $S^m(\boldsymbol{\lambda})$ over all possible break dates. These tests have the same properties as their counterparts in the one break case: MS^m is effective under small breaks while $S^m(\hat{\boldsymbol{\lambda}})$ is effective under large breaks.

Let us detect the number of breaks equal to $\hat{m}$ (e.g. by the sequential procedures from Kejriwal and Perron (2010), Sobreira and Nunes (2012) and Harvey *et al.* (2010)). The extension of the

[4]This occurs only in the unobserved component representation due to imposing the additional noise component under the alterbative hypothesis.

strategy in (5) can be constructed as follows:

$$A^m(\hat{m}) : \text{Reject } H_0 \text{ if } \begin{cases} MS^m > cv_{MS^m}, & \hat{m} = 0 \\ \{MS^m > cv_{MS^m} \text{ и } S(\hat{\boldsymbol{\lambda}}) > cv_{S(\hat{\boldsymbol{\lambda}})}\}, & 0 < \hat{m} < m \\ S(\hat{\boldsymbol{\lambda}}) > cv_{S(\hat{\boldsymbol{\lambda}})}, & \hat{m} = m \end{cases}, \tag{7}$$

where m is the maximum allowable number of breaks, $\hat{m}$ is the estimated number of breaks, and cv_{MS^m} and $cv_{S(\hat{\boldsymbol{\lambda}})}$ are corresponding critical values.

Table 3 represents the simulation results for the two break case, $m = 2$, with a DGP similar to the previous section for $\beta_2 = \gamma\beta_1$, $\gamma \in \{0.5, 1, -0.5, -1\}$, the number of replications is 1,000. We consider only the MS^2, $S^2(\hat{\boldsymbol{\lambda}})$ and $A^2(\hat{m})$ tests, where the number of breaks, $\hat{m}$, is selected by the procedure from Sobreira and Nunes (2012) (the analogue of generalizations of Kejriwal and Perron (2010), but for the Harvey *et al.* (2009) test, not for the Perron and Yabu (2009)), because the Kejriwal and Perron (2010) procedure inherits the poor properties of the Perron and Yabu (2009) test under the alternative hypothesis of nonstationarity. The critical values for the $S^2(\hat{\boldsymbol{\lambda}})$ test are provided in Carrion-i-Silvestre and Sansó-i-Rosselló (2005), and the critical values for the MS^2 and MS^3 are provided in Table 1.

As expected, the MS^2 test has serious size distortion under large breaks, especially for $\gamma = \pm 1$. The size of the $S^2(\hat{\boldsymbol{\lambda}})$ is very undersized, although it approaches a nominal size with increasing break magnitudes. The $A^2(\hat{m})$ test saves the high power under small breaks and controls size under large breaks. Note that for breaks with opposite signs, it is more difficult to detect the presence of breaks using the sequential procedure, and the power of $A^2(\hat{m})$ even for moderate breaks is similar to the power of MS^2, although there are no serious size distortions.

4 Conclusion

The magnitude of structural breaks can have a large impact on statistical inference. Within the context of the investigation of the integrating order, the results may depend on the size of a structural break. Recently, procedures have been developed for determining the significance of breaks regardless of the order of integration of the series. These tests can be used as a pre-tests for the presence of breaks, and then that information can then be used to test the unit root or stationarity hypothesis.

In this paper, we used a pre-test for breaks in the context of stationarity testing and proposed the decision rules based on the use of multiple tests. These decision rules show good finite samples properties, and in the absence of a priori information about the magnitude of the breaks, they can be used as a risk-averse strategy for testing the stationarity of the series.

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Table 1. ξ level critical values for the MS^2

ξ	Model 0	Model 0t	Model I	Model II
1%	0.064	0.039	0.048	0.023
5%	0.047	0.030	0.038	0.020
10%	0.041	0.027	0.033	0.018

Table 2. The size and power of the tests, 1 break

β_1	$q=0$				$q=0.2$			
	$S(\hat{\lambda})$	MS	$A(t_{HLT})$	$A(t_{PY})$	$S(\hat{\lambda})$	MS	$A(t_{HLT})$	$A(t_{PY})$
$\lambda=0.3$								
0.00	0.013	0.053	0.053	0.051	0.474	0.650	0.639	0.530
0.01	0.014	0.053	0.053	0.048	0.479	0.648	0.636	0.532
0.02	0.016	0.057	0.057	0.040	0.473	0.641	0.630	0.522
0.04	0.014	0.052	0.050	0.016	0.478	0.646	0.626	0.515
0.10	0.021	0.055	0.041	0.021	0.541	0.678	0.624	0.546
0.20	0.024	0.055	0.028	0.024	0.581	0.702	0.605	0.581
0.40	0.026	0.058	0.026	0.026	0.608	0.716	0.608	0.608
1.00	0.037	0.079	0.037	0.037	0.627	0.731	0.627	0.627
2.00	0.054	0.168	0.054	0.054	0.664	0.802	0.664	0.664
$\lambda=0.5$								
0.00	0.013	0.053	0.053	0.051	0.474	0.650	0.639	0.530
0.01	0.010	0.037	0.037	0.033	0.476	0.644	0.631	0.528
0.02	0.007	0.025	0.025	0.013	0.462	0.634	0.622	0.517
0.04	0.008	0.020	0.018	0.009	0.443	0.610	0.586	0.480
0.10	0.007	0.019	0.012	0.007	0.444	0.568	0.505	0.447
0.20	0.010	0.019	0.011	0.010	0.454	0.556	0.467	0.454
0.40	0.009	0.019	0.009	0.009	0.473	0.561	0.473	0.473
1.00	0.025	0.039	0.025	0.025	0.514	0.600	0.514	0.514
2.00	0.050	0.129	0.050	0.050	0.605	0.753	0.605	0.605
$\lambda=0.7$								
0.00	0.013	0.053	0.053	0.051	0.474	0.650	0.639	0.530
0.01	0.013	0.054	0.054	0.047	0.475	0.643	0.631	0.527
0.02	0.014	0.050	0.050	0.035	0.471	0.640	0.626	0.522
0.04	0.015	0.049	0.047	0.017	0.479	0.641	0.622	0.517
0.10	0.021	0.053	0.039	0.021	0.530	0.671	0.618	0.535
0.20	0.023	0.053	0.030	0.023	0.575	0.698	0.600	0.575
0.40	0.023	0.051	0.023	0.023	0.600	0.708	0.600	0.600
1.00	0.036	0.078	0.036	0.036	0.623	0.725	0.623	0.623
2.00	0.048	0.170	0.048	0.048	0.654	0.799	0.654	0.654

Table 3. Размер и мощность тестов, 2 сдвига

β_1	$q=0$			$q=0.2$		
	$S^2(\hat{\boldsymbol{\lambda}})$	MS^2	$A^2(\hat{m})$	$S^2(\hat{\boldsymbol{\lambda}})$	MS^2	$A^2(\hat{m})$
			$\gamma=0.5$			
0.00	0.005	0.056	0.056	0.179	0.472	0.442
0.01	0.005	0.053	0.053	0.197	0.484	0.452
0.02	0.004	0.052	0.046	0.199	0.497	0.448
0.04	0.005	0.027	0.016	0.189	0.484	0.414
0.10	0.001	0.025	0.017	0.201	0.484	0.366
0.20	0.003	0.020	0.011	0.179	0.422	0.304
0.40	0.000	0.013	0.000	0.132	0.345	0.131
1.00	0.005	0.042	0.005	0.154	0.388	0.154
2.00	0.018	0.104	0.018	0.265	0.525	0.265
			$\lambda=1$			
0.00	0.005	0.056	0.056	0.179	0.472	0.442
0.01	0.005	0.055	0.052	0.199	0.479	0.445
0.02	0.004	0.044	0.044	0.200	0.500	0.446
0.04	0.005	0.014	0.009	0.177	0.467	0.375
0.10	0.005	0.023	0.018	0.165	0.458	0.345
0.20	0.002	0.020	0.003	0.176	0.391	0.231
0.40	0.000	0.017	0.000	0.126	0.344	0.123
1.00	0.005	0.058	0.005	0.167	0.426	0.167
2.00	0.027	0.217	0.027	0.350	0.696	0.350
			$\gamma=-0.5$			
0.00	0.005	0.056	0.056	0.179	0.472	0.442
0.01	0.007	0.052	0.052	0.199	0.488	0.458
0.02	0.008	0.061	0.060	0.201	0.510	0.474
0.04	0.004	0.028	0.028	0.186	0.454	0.428
0.10	0.007	0.019	0.019	0.190	0.449	0.418
0.20	0.003	0.025	0.025	0.152	0.372	0.366
0.40	0.001	0.012	0.006	0.151	0.357	0.275
1.00	0.004	0.041	0.002	0.157	0.388	0.141
2.00	0.020	0.104	0.020	0.272	0.519	0.269
			$\gamma=-1$			
0.00	0.005	0.056	0.056	0.179	0.472	0.442
0.01	0.009	0.053	0.053	0.197	0.487	0.463
0.02	0.009	0.052	0.052	0.186	0.498	0.468
0.04	0.000	0.017	0.017	0.178	0.455	0.428
0.10	0.005	0.017	0.017	0.178	0.409	0.404
0.20	0.002	0.024	0.024	0.151	0.366	0.366
0.40	0.000	0.014	0.014	0.142	0.345	0.344
1.00	0.007	0.058	0.026	0.164	0.421	0.294
2.00	0.027	0.216	0.027	0.359	0.687	0.359