On GLS-detrending for deterministic seasonality testing

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Abstract

In this paper we propose tests based on GLS-detrending for testing the null hypothesis of deterministic seasonality. Unlike existing tests for deterministic seasonality, our tests do not suffer from asymptotic size distortions under near integration. We also investigate the behavior of the proposed tests when the initial condition is not asymptotically negligible.
Key words: Stationarity tests, KPSS test, seasonality, seasonal unit roots, deterministic seasonality, size distortion, GLS-detrending.
JEL: C12, C22.

1 Introduction

Deterministic seasonality describes the behavior of a time series in which the unconditional means change in different seasons of the year. One way to record this is a seasonal dummies representation. On the other hand, the presence of a seasonal unit root in the data could distort the seasonal correction procedure for the time series, so dividing the processes using deterministic seasonality and seasonal unit root processes is important in the time series analysis. Most of the work, following Hylleberg *et al.* (1990) (hereinafter HEGY), focuses on testing for seasonal unit roots at different frequencies against the alternative that all the roots are less than one. This procedure is related to the testing of the unit root (at zero frequency) against the alternative of stationarity in the nonseasonal case.

Similar to the nonseasonal case (the stationarity test against the alternative hypothesis of a unit root, see Kwiatkowski *et al.* (1992)) the procedures for deterministic seasonality testing against the case of the presence of at least one seasonal unit root has also been developed. The problem of deterministic seasonality testing was first considered by Canova and Hansen (1995). Taylor (2003*a*) analyzed a more general formulation in the context of the construction of locally

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mean most powerful invariant (LMMPI) tests. Taylor (2003*b*) also shows that under the presence of unattended unit roots, i.e. if the null hypothesis is tested against the alternative of a specific number of unit roots, but actually the series contains a number of additional roots at other frequencies, then the test statistic under the null will be $O_p((TS_T)^{-1})$. This means the test will be conservative (and size distortion will increase as the sample size increases). To solve this problem, the author proposed the use of pre-filtered data similar to HEGY test. Kurozumi (2002) investigated the limiting properties of the Canova and Hansen test. The author has found the limiting distributions of tests by using the Fredholm approach (see Tanaka (1996)), and also has shown that the power of the tests depends not only on the local parameter, c, but also on the reciprocal of the spectral density of the stationary component of the time series at frequency π or $\pi/2$.

There are alternative approaches to testing for deterministic seasonality. In Caner (1998), in contrast to the non-parametric approach proposed by Canova and Hansen (1995) and Taylor (2003*a*), the parametric autocorrelation correction of errors according to the Leybourne and McCabe (1994) principle is used (but under stronger assumptions). More precisely, Caner (1998) uses the residuals from regression not only on the deterministic component, but also on a sufficient number of lagged dependent variables. In this case, there is no need to construct a nonparametric estimator of long-run variance and a conventional variance estimate of pre-whitened data should be used. The distributions of the test statistics coincide with the results obtained in Canova and Hansen (1995) and Taylor (2003*a*). Another approach for deterministic seasonality testing was developed by Tam and Reinsel (1997, 1998) in the context of testing for the unit root in the MA component of a time series. For comparisons of this approach with the Canova and Hansen tests, see Ghysels and Osborn (2001, Sections 2.4.3, 2.4.4).

One problem is that under near integration all seasonal unit root tests at different frequencies will have an asymptotic size equal to unity (see Müller (2005) for the nonseasonal case). One solution to this problem (in the nonseasonal case) has been proposed by Harris *et al.* (2007) (henceforth HLM), where the authors used a (quasi) GLS-detrending to construct the test statistics. Here we generalize their approach to a seasonal case and we test the hypothesis of deterministic seasonality (local to a seasonal unit root) against the alternative of a seasonal unit root.

This paper is organized as follows. Section 2 describes the data generating process (DGP) and assumptions about errors and initial conditions (the assumptions about the initial condition follow from Harvey *et al.* (2008) (henceforth HLT)). In Section 3 we propose the procedure of seasonal GLS-detrending for stationarity test statistics. The Monte-Carlo simulation results (asymptotic and finite sample) are given in Section 4. The results are formulated in the Conclusion.

2 Model

Consider quarterly DGP such that

$$y_{4n+s} = \mu_{4n+s} + u_{4n+s}, \; s = -3, \ldots, 0, \; n = 1, \ldots, N, \quad (1)$$

$$a(L) u_{4n+s} = \varepsilon_{4n+s}, \; s = -3, \ldots, 0, \; n = 2, \ldots, N, \quad (2)$$

where $a(L) = 1 - \sum_{j=1}^{4} a_j L^j$ is a fourth order autoregressive polynomial, L is the lag operator such that $L^{4j+k} y_{4n+s} = y_{4(n-j)+s-k}$, $T = 4N$ is the number of observations (N is the span in years of the sample data). The errors ε_{4n+s} are assumed to be a zero mean process, the long-run variance of which is bounded and strongly positive at zero and seasonal spectral frequencies, $\omega_k = 2\pi k/4$, $k = 0, 1, 2$.

The deterministic component $\mu_{4n+s} = \mu_t$ is defined as a linear combination of spectral indicator variables, corresponding to the zero and seasonal frequencies, $z_{t,0} = 1$, $z_{t,1} = (\cos[2\pi t/4], \sin[2\pi t/4])'$ and $z_{t,2} = (-1)^t$. Define the vector $Z_t = (z_{t,0}, z'_{t,1}, z_{t,2})'$ and the deterministic component $\mu_t = d'_\xi Z_{t,\xi}$ for three possible cases, $\xi = 1, \ldots, 3$ (see Smith and Taylor (1998)). The first case corresponds to the constants at zero and seasonal frequencies, $Z_{t,1} = Z_t$, the second case also allows for the trend at zero frequency, $Z_{t,2} = (Z'_t, z_{t,0}t)'$, the third case allows for the trend at zero and seasonal frequencies, $Z_{t,3} = (Z'_t, tZ'_t)'$.

The polynomial $a(L)$ can be factorized as $\prod_{k=0}^{2} \omega_k(L) = (1-a_0L)(1-2\beta_1 L+(a_1^2+\beta_1^2)L^2)(1-a_2L)$. We are interested in testing for deterministic seasonality (local to the seasonal unit root), in other words to test the hypothesis $H_0 : c_i \geq \bar{c}_i > 0$ for all i in $a_i = 1-c_i/T$, against the alternative about the unit root at least one of the frequencies, i.e. $H_1 : c_i = 0$ for at least one i, where $\bar{c}_i$ is the minimal amount of mean reversion for the specific frequency under the null hypothesis. The null hypothesis H_0 can be partitioned as $H_0 = \bigcap_{k=0}^{2} H_{0,c_k}$, where $H_{0,c_i} : a_i = 1 - c_i/T$, $i = 0, 2$ and $H_{0,c_1} : a_1 = 1 - c_1/T$, $\beta_1 = 0$. In other words, testing the null hypothesis of stationarity (local to the unit root) at zero frequency, $\omega_0 = 0$, against the alternative of a unit root at this frequency is equivalent to testing $H_{0,c_0} : c_0 \geq \bar{c}_0 > 0$ against $H_{0,c_0} : c_1 = 0$. Similarly, testing the null hypothesis for stationarity (local to unit root) at the Nyquist frequency, $\omega_2 = \pi$, against the alternative of a unit root at this frequency is equivalent to testing $H_{0,c_2} : c_2 \geq \bar{c}_2 > 0$ against $H_{2,c_2} : c_2 = 0$, and testing the null hypothesis of stationarity (local to unit root) at seasonal harmonic frequencies, $(\pi/2, 3\pi/2)$, against the alternative of unit roots at these frequencies is equivalent to testing $H_{0,c_1} : c_1 \geq \bar{c}_1 > 0$ against $H_{1,c_1} : c_1 = 0$.

This approach differs to the usual testing for deterministic seasonality, where either the local asymptotic behavior is not considered or parameters related to the signal-to-noise ratio are assumed to be local (see Taylor (2003*a*)). The reason for considering near integration is the same as in Müller (2005): it explains the increasing size in finite samples for highly autocorrelated stationary series. As demonstrated in Müller (2005) in the context of nonseasonal models, the conventional KPSS test with the bandwidth parameter in the long-run variance estimator increasing at a slower rate than the length of the sample leads to an asymptotic size equal to unity under the null hypothesis of near integration. It is easy to show that the same problem arises in the seasonal models, if we use the tests of Canova and Hansen (1995), Taylor (2003*a*) and Taylor (2003*b*), *inter alia*. One way to solve this problem will be considered in the next section where we extend the HLM test to a seasonal case.

Also we set the initial condition u_i, $i = 1, \ldots, 4$, according to Assumption 1 (see HLT).

Assumption 1 *Under H_0 with $c < 0$, the initial conditions are generated according to*

$$u_i = \alpha_i \sqrt{\omega_\varepsilon^2/(1-\rho_N^2)},\ i = 1, \ldots, 4, \tag{3}$$

where $\rho_N = 1 - c/N$ and $\alpha_i \sim IN(\mu_{\alpha,i}\mathbb{I}(\sigma_\alpha^2 = 0), \sigma_\alpha^2)$, $i = 1, \ldots, 4$, independent of ε_{4n+s}, $s = -3, \ldots, 0$, $n = 2, \ldots, N$. For $c = 0$, under H_1, the initial conditions can be set equal to zero, $u_i = 0$, $i = 1, \ldots, 4$, without loss of generality, due to the exact similarity of the tests to the initial conditions in this case.

In this assumption α_i controls the magnitude of the initial condition in season i relative to the magnitude of the standard deviation of a stationary seasonal AR(1) process with parameter ρ_N and innovation long-run variance ω_ε^2. The form given for the u_i allow the initial conditions to be either random and of $O_p(N^{1/2})$, or fixed and of $O(N^{1/2})$ depending on the value of variance σ_α^2 (> 0 or 0, respectively).

3 Deterministic seasonality testing based on GLS-detrending

HLM proposed the following test using a (quasi) GLS-detrended series. More precisely, let $\tilde{u}_t^{\xi}$ be the residuals from regression $y_{\bar{c}} = y_t - \bar{\rho}_T y_{t-1}$ on $Z_{i,\bar{c}} = z_t - \bar{\rho}_T z_{t-1}$, $t = 2, \ldots, T$, where $z_t = 1$ in constant case ($\xi = \mu$) and $z_t = (1, t)'$ in trend case ($\xi = \tau$) and $\bar{\rho}_T = 1 - \bar{c}/T$. Then the $S^{\xi}(\bar{c})$ test is constructed as following:

$$S^{\xi}(\bar{c}) = \frac{T^{-2} \sum_{t=2}^{T} (\sum_{j=2}^{t} \tilde{u}_j^i)^2}{\hat{\omega}^2}, \tag{4}$$

where the kernel based long-run variance estimator $\hat{\omega}^2$ is calculated by using GLS-detrended residuals $\tilde{u}_t^i$.

For a seasonal time series consider the following GLS-transformation (see also Rodrigues and Taylor (2007) in the context of seasonal unit root testing), by using a vector $\mathbf{c} = (\bar{c}_0, \bar{c}_1, \bar{c}_2)$. Let the series y_c and $\mathbf{Z}_{\xi,\mathbf{c}}$ be defined as

$$\begin{aligned} y_{\mathbf{c}} &= (\Delta_{\mathbf{c}} y_{S+1}, \ldots, \Delta_{\mathbf{c}} y_T)' \\ \mathbf{Z}_{\xi,\mathbf{c}} &= (\Delta_{\mathbf{c}} Z_{S+1,\xi}, \ldots, \Delta_{\mathbf{c}} Z_{T,\xi})', \end{aligned}$$

where

$$\Delta_{\mathbf{c}} = 1 - \sum_{j=1}^{S} a_j^{\mathbf{c}} L^j = \left(1 - \left(1 - \frac{\bar{c}_0}{T}\right) L\right) \left(1 + \left(1 - \frac{\bar{c}_2}{T}\right) L\right) \left(1 + \left(1 - \frac{\bar{c}_1}{T}\right)^2 L^2\right)$$

(respectively, $\Delta_{\mathbf{0}} = 1 - L^S$). GLS-detrended series are the OLS-residuals from regression y_c on $\mathbf{Z}_{\xi,\mathbf{c}}$. Define these residuals as $\tilde{u}_{t,\xi}$.

Before constructing the test statistics, we note that one of the basic principles of unit root testing in a seasonal time series is that when we test for a unit root at a specific frequency, the data should be prefiltered to reduce the order of integration at each of the remaining (unattended) frequencies by one (see HEGY and Taylor (2003*b*)). If we are testing the stationarity against the presence of a unit root at zero frequency it is necessary to use the data $y_{0,t} = (1+L)(1+L^2)y_t$ instead of y_t, and then perform GLS-detrending. Accordingly for testing the stationarity at Nyquist frequency it is necessary to use the data $y_{2,t} = (1-L)(1+L^2)y_t$, and for testing the stationarity at the seasonal harmonic frequencies it is necessarily to use the data $y_{1,t} = (1-L)(1+L)y_t$. We define the corresponding GLS-detrended residuals $\tilde{u}_{t,\xi}^j$ depending on $j = 0, 1, 2$, corresponding to zero frequency, seasonal harmonic frequencies and the Nyquist frequency, respectively.

Thus, a final test can be written as follows:

$$S_{\xi,j}(\mathbf{c}) = T^{-2} tr \left[(C_j' \hat{\Omega}_Z C_j)^{-1} C_j' \sum_{t=1}^{T} \left(\sum_{s=1}^{t} \tilde{u}_{s,\xi}^j Z_s \right) \left(\sum_{s=1}^{t} \tilde{u}_{s,\xi}^j Z_s' \right) C_j \right], \tag{5}$$

where $\hat{\Omega}_Z$ is a consistent long-run variance estimator of $Z_t \varepsilon_t$, where

$$\hat{\Omega}_Z = \sum_{l=-T+1}^{T-1} k\left(\frac{j}{S_T}\right) \hat{\Gamma}(l), \tag{6}$$

with the bandwidth parameter $S_T \to \infty$, $S_T/T^{1/2} \to 0$ and the autocovariance estimator

$$\hat{\Gamma}(l) = T^{-1} \sum_{s=l+1}^{T} \tilde{u}_{s,\xi}^j Z_t \tilde{u}_{s-l,\xi}^j Z_{s-l}', \ \hat{\Gamma}(-l) = \hat{\Gamma}(l), \ l \geq 0. \tag{7}$$

The matrix C_j depends on the frequencies at which we test the stationarity. For zero frequency C_0 is the first column of identity matrix $\mathbf{I}_4$, for Nyquist frequency C_2 is the fourth column of $\mathbf{I}_4$, for seasonal harmonic frequencies C_1 is the matrix consisting of the second and third columns of $\mathbf{I}_4$. For testing the stationarity at all seasonal frequencies the C_{12} matrix contains the second, third and forth columns, and for testing the stationarity at all seasonal and nonseasonal frequencies C_{012} is the identity matrix $\mathbf{I}_4$.

The following proposition gives the limiting distributions of all test statistics to test the stationarity (local to the seasonal unit root) at specific frequencies under the null and alternative hypotheses.

Proposition 1 *Let* $\{y_{Sn+s}\}$ *be generated as* (1)-(2) *under Assumption 1. For* $i = 1, \ldots, 4$ *we define*

$$K_{ic}(r) = \begin{cases} W_i(r), & \text{if } c = 0, \\ \bar{\alpha}_i(e^{rc} - 1)(-2c)^{-1/2} + W_{ic}(r), & \text{if } c < 0, \end{cases}$$

where $W_i(r)$, $i = 1, \ldots, 4$, *are independent standard Wiener processes,* $W_{ic}(r)$, $i = 1, \ldots, 4$, *are independent Ornstein-Uhlenbeck processes, and the spectral magnitudes* $\bar{\alpha}_i$ *are defined as*

$$\begin{aligned} \bar{\alpha}_1 &= (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)/2, \\ \bar{\alpha}_2 &= (-\alpha_1 + \alpha_2 - \alpha_3 + \alpha_4)/2, \\ \bar{\alpha}_3 &= (\alpha_4 - \alpha_2)/\sqrt{2}, \\ \bar{\alpha}_4 &= (\alpha_3 - \alpha_1)/\sqrt{2}. \end{aligned}$$

Then

$$S_j(\bar{c}_j) \Rightarrow \int_0^1 H_{jc,\bar{c}_j}(r)^2 dr, \; j = 0, 2, \tag{8}$$

$$S_1(\bar{c}_1) \Rightarrow \int_0^1 (H_{1c,\bar{c}_1}(r) + H_{3c,\bar{c}_1}(r))^2 dr, \tag{9}$$

$$S_{12}(\bar{c}_1) \Rightarrow \int_0^1 (H_{1c,\bar{c}_1}(r) + H_{2c,\bar{c}_2}(r) + H_{3c,\bar{c}_1}(r))^2 dr, \tag{10}$$

$$S_{012}(\bar{c}_1) \Rightarrow \int_0^1 (H_{0c,\bar{c}_0}(r) + H_{1c,\bar{c}_1}(r) + H_{2c,\bar{c}_2}(r) + H_{3c,\bar{c}_1}(r))^2 dr, \tag{11}$$

where

$$H_{ic,\bar{c}_j}(r) = K_{ic}(r) + \bar{c}_j \int_0^r K_{ic}(s) ds - r \left[K_{ic}(1) + \bar{c}_j \int_0^1 K_{ic}(s) ds \right],$$

$i = 1, \ldots, 4$, *for* $j = 0$, $\xi = 0$ *and for* $j = 1, 2$, $\xi = 0, 1$, *and*

$$H_{ic,\bar{c}_j}(r) = H_{ic,\bar{c}_j}(r) - 6r(1 - r) \int_0^1 H_{ic,\bar{c}_j}(s) ds,$$

$i = 1, \ldots, 4$, *for* $j = 0$, $\xi = 1, 2$ *and for* $j = 1, 2$, $\xi = 3$, *and* $\Rightarrow$ *denotes weak convergence.*

The proof of this proposition follows from HLM, Taylor (2003a) and HLT and is omitted for brevity. Similar to HLM, it can be shown that for $c_j = \bar{c}_j$ the limiting distribution of the S_j test coincides with the results obtained in Canova and Hansen (1995) and Taylor (2003a) (under the null hypothesis) and does not depend on initial conditions. Similar results will be occurred for $c_1 = \bar{c}_1$ and $c_2 = \bar{c}_2$ for the S_{12} test and for $c_0 = \bar{c}_0$, $c_1 = \bar{c}_1$ and $c_2 = \bar{c}_2$ for the S_{012} test. Critical values are given in Tables 1-3.

Also it can be noted that the limiting distributions depend not on the values of the initial conditions α_i, but on the so-called spectral initial conditions $\bar{\alpha}_i$ in the terminology of HLT. Hence for some given nonzero initial conditions, their linear combination may be zero and some tests will not depend on their magnitudes.

4 Monte-Carlo simulations

4.1 Asymptotic results

In this subsection, we analyze the asymptotic behavior of S_0, S_2, S_1, S_{12} and S_{012} tests under various magnitudes for the initial conditions[1]. Figure 1 shows results for the case of a fixed initial condition ($\sigma_\alpha^2 = 0$ in Assumption 1), while Figure 2 shows results for the case of a random initial condition ($\sigma_\alpha^2 > 0$ in Assumption 1). Everywhere we consider a model with seasonal constants and a nonseasonal trend (since this model is most often used in practice) and use $\bar{c}_0 = 13.5$, $\bar{c}_2 = 7$, $\bar{c}_1 = 3.75$.

Parts (a), (b) and (c) in Figure 1 represent, respectively, asymptotic size ($a(L) = 1 - (1 - c/N)L^4$, $c > 0$) and power ($c = 0$) of S_0, S_2 and S_1 tests for various magnitudes of initial conditions $\bar{\alpha}_1$, $\bar{\alpha}_2$ and $\bar{\alpha}_3 = \bar{\alpha}_4$, respectively. The magnitudes of the initial conditions $\bar{\alpha}_i = |\mu| = \{0, 2, 4, 6\}$, $i = 1, \ldots, 4$ are used for all the tests being considered. Parts (d), (e) and (f) represent results for the S_{12} test under: $\bar{\alpha}_2 = |\mu|$, $\bar{\alpha}_3 = \bar{\alpha}_4 = 0$; $\bar{\alpha}_2 = 0$, $\bar{\alpha}_3 = \bar{\alpha}_4 = |\mu|$; $\bar{\alpha}_2 = \bar{\alpha}_3 = \bar{\alpha}_4 = |\mu|$, respectively. Parts (g), (h) and (i) represent results for the S_{012} test under: $\bar{\alpha}_1 = |\mu|$, $\bar{\alpha}_2 = \bar{\alpha}_3 = \bar{\alpha}_4 = 0$; $\bar{\alpha}_1 = \bar{\alpha}_2 = 0$, $\bar{\alpha}_3 = \bar{\alpha}_4 = |\mu|$; $\bar{\alpha}_1 = \bar{\alpha}_2 = \bar{\alpha}_3 = \bar{\alpha}_4 = |\mu|$, respectively. Corresponding results for the case of random initial conditions are given in Figure 2 for $\sigma_\alpha^2 = |\mu| = \{0, 2, 4, 6\}$, $i = 1, \ldots, 4$. In all cases critical values are obtained at $c_j = \bar{c}_j$.

The size curves of S_0, S_2 and S_1 tests are tangent to each other for different initial conditions at points $c_j = \bar{c}_j$, which confirms the invariance with respect to the initial conditions of each of the tests at $c = \bar{c}_j$. When $c < \bar{c}_j$ the size of each test grows as the corresponding spectral initial condition is increased, although the size distortion is less pronounced for the S_0 test (intuitively because a larger value is used for $\bar{c}_j$). For $c > \bar{c}_j$ the size of the test is lower that the nominal one, except for the S_1 test with $|\mu| = 6$ (because a very small value of $\bar{c}_j$ is used). Similar behavior is observed for S_{12} and S_{012}.

For the case of random initial conditions, the size distortions for $c < \bar{c}_j$ are smaller that in case of fixed initial conditions, but larger than for $c > \bar{c}_j$.

[1]Here and in the following subsection results are obtained by simulations of the limiting distributions in Proposition 1, approximating the Wiener process using $i.i.d.N(0, 1)$ random variates and with integrals approximated by normalized sums of 1,000 steps, with 50,000 replications.

4.2 Finite sample comparisons

In this subsection we investigate the finite sample behavior of tests based on DGP (1)-(2) and Assumption 1. Results are presented for $T = 300$ with 10,000 replications and, without loss of generality, with the absence of the deterministic term. The long-run variance estimator (6) is constructed by using a quadratic spectral window and automatic bandwidth selection based on AR(1) approximation (see Andrews (1991)).

Figures 3-4 show the size and power of the tests (the notation is the same as in Section 4.1) for the cases of fixed and random initial conditions, respectively. The results are similar to the asymptotic case although the power becomes slightly lower. However all tests control size well under the very close to unity autoregression parameter. To increase the power, it can sacrifice this size control and use a higher $\bar{c}_j$, which will lead to control the size at a higher c. One alternative is to use parameters $\bar{c}_j$ such that the asymptotic size of S_0, S_2 and S_1 tests (e.g., in case of a seasonal constant and nonseasonal trend) is equal to 0.5 at $c_0 = 13.5$, $c_2 = 7$, $c_1 = 3.75$. These new values of $\bar{c}_j$ are equal to $\bar{c}_0 = 29.55$, $\bar{c}_2 = 20.05$ and $\bar{c}_1 = 10.7$ (in case of seasonal trend $\bar{c}_1 = 17.75$ and $\bar{c}_2 = 29.55$). Corresponding values of power in finite samples ($T = 300$) for S_0, S_2, S_1, S_{01} and S_{012} tests will then be equal to 0.40, 0.58, 0.58, 0.57 and 0.621, respectively. Of course, the researcher can choose any values of $\bar{c}_j$ for his desired trade-off between size and power.

5 Conclusion

In this paper, we consider GLS-detrendng in the context of deterministic seasonality testing (local to a seasonal unit root) against the alternative of a seasonal unit root. Unlike existing tests, the proposed tests do not suffer from asymptotic size distortions in the local to unity autoregression parameters and, at the same time, have a well-controlled size for a given level of (seasonal) mean reversion. The behavior of the proposed tests was also analyzed for fixed and random initial conditions of various magnitudes.

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Table 1. Critical values at the ξ significance level, Case 1 (seasonal constant with no trend), $\bar{c}_0 = 7$, $\bar{c}_2 = 7$, $\bar{c}_1 = 3.75$

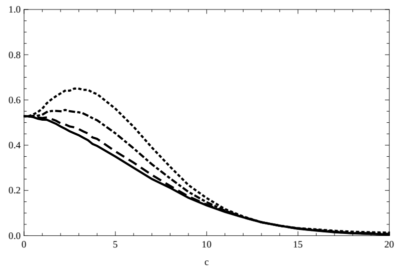
	T	$\xi = 0.10$	$\xi = 0.05$	$\xi = 0.01$
$T = 152$	S_0	0.303	0.374	0.501
	S_2	0.301	0.372	0.509
	S_1	0.506	0.569	0.679
	S_{12}	0.677	0.738	0.834
	S_{012}	0.825	0.878	0.966
$T = 300$	S_0	0.323	0.414	0.595
	S_2	0.323	0.413	0.599
	S_1	0.554	0.646	0.828
	S_{12}	0.756	0.854	1.036
	S_{012}	0.932	1.026	1.200
$T = 600$	S_0	0.335	0.433	0.668
	S_2	0.337	0.436	0.667
	S_1	0.582	0.694	0.922
	S_{12}	0.797	0.921	1.158
	S_{012}	0.993	1.121	1.359
$T = 900$	S_0	0.342	0.449	0.694
	S_2	0.342	0.445	0.687
	S_1	0.591	0.715	0.987
	S_{12}	0.812	0.946	1.223
	S_{012}	1.018	1.158	1.436

Table 2. Critical values at the ξ significance level, Case 2 (seasonal constant, nonseasonal trend), $\bar{c}_0 = 13.5$, $\bar{c}_2 = 7$, $\bar{c}_1 = 3.75$

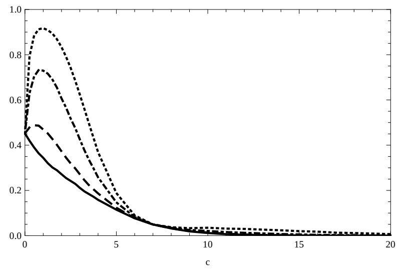
		$\xi = 0.10$	$\xi = 0.05$	$\xi = 0.01$
$T = 152$	S_0	0.104	0.119	0.151
	S_2	0.302	0.371	0.502
	S_1	0.506	0.566	0.670
	S_{12}	0.678	0.739	0.838
	S_{012}	0.783	0.844	0.942
$T = 300$	S_0	0.110	0.131	0.177
	S_2	0.323	0.412	0.598
	S_1	0.554	0.644	0.823
	S_{12}	0.757	0.855	1.037
	S_{012}	0.849	0.949	1.133
$T = 600$	S_0	0.114	0.139	0.193
	S_2	0.337	0.436	0.666
	S_1	0.582	0.694	0.920
	S_{12}	0.797	0.921	1.159
	S_{012}	0.881	1.007	1.248
$T = 900$	S_0	0.116	0.140	0.197
	S_2	0.342	0.445	0.686
	S_1	0.591	0.714	0.987
	S_{12}	0.812	0.946	1.223
	S_{012}	0.893	1.029	1.312

Table 3. Critical values at the ξ significance level, Case 3 (seasonal constant, seasonal trend), $\bar{c}_0 = 13.5$, $\bar{c}_2 = 13.5$, $\bar{c}_1 = 8.65$

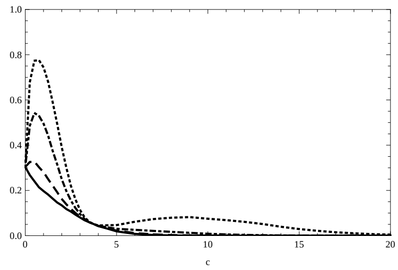
		$\xi = 0.10$	$\xi = 0.05$	$\xi = 0.01$
$T = 152$	S_0	0.104	0.120	0.151
	S_2	0.103	0.119	0.150
	S_1	0.193	0.208	0.237
	S_{12}	0.268	0.284	0.314
	S_{012}	0.343	0.361	0.404
$T = 300$	S_0	0.110	0.131	0.177
	S_2	0.110	0.130	0.174
	S_1	0.196	0.219	0.264
	S_{12}	0.275	0.300	0.346
	S_{012}	0.347	0.372	0.416
$T = 600$	S_0	0.114	0.139	0.193
	S_2	0.115	0.139	0.192
	S_1	0.203	0.231	0.288
	S_{012}	0.285	0.317	0.382
	S_{012}	0.360	0.393	0.457
$T = 900$	S_0	0.116	0.140	0.197
	S_2	0.116	0.142	0.201
	S_1	0.205	0.236	0.301
	S_{12}	0.289	0.324	0.393
	S_{012}	0.367	0.402	0.475



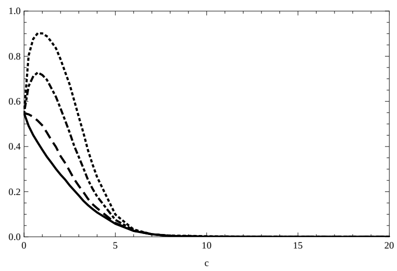
(a) S_0: $\bar{\alpha}_1 = |\mu|$



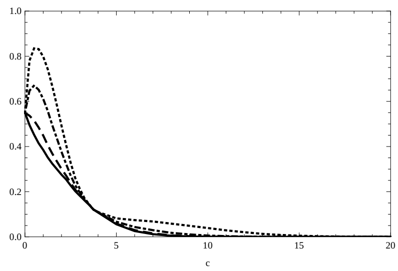
(b) S_2: $\bar{\alpha}_2 = |\mu|$



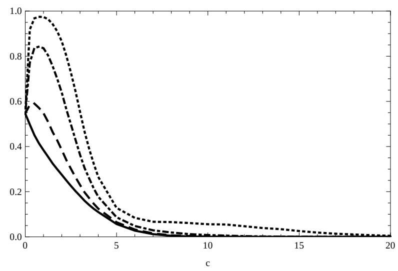
(c) S_1: $\bar{\alpha}_3 = \bar{\alpha}_4 = |\mu|$



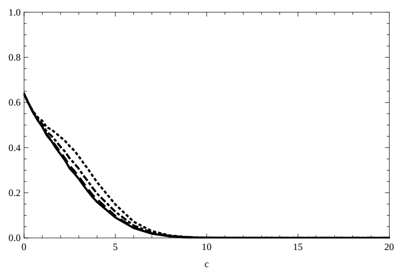
(d) S_{12}: $\bar{\alpha}_2 = |\mu|$, $\bar{\alpha}_3 = \bar{\alpha}_4 = 0$



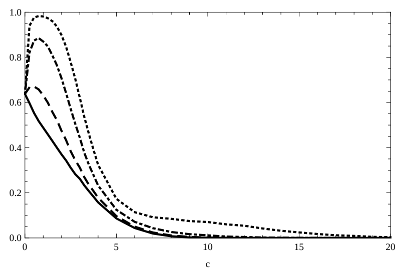
(e) S_{12}: $\bar{\alpha}_2 = 0$, $\bar{\alpha}_3 = \bar{\alpha}_4 = |\mu|$



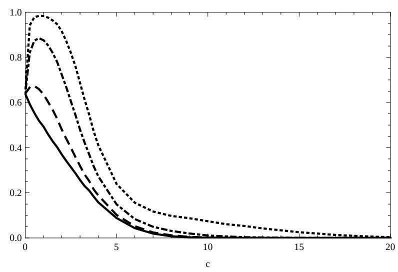
(f) S_{12}: $\bar{\alpha}_2 = \bar{\alpha}_3 = \bar{\alpha}_4 = |\mu|$



(g) S_{012}: $\bar{\alpha}_1 = |\mu|$,
$\bar{\alpha}_2 = \bar{\alpha}_3 = \bar{\alpha}_4 = 0$



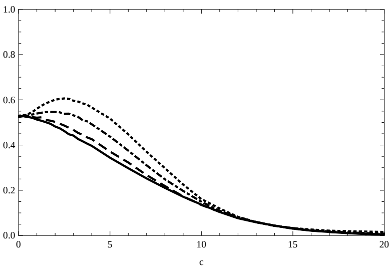
(h) S_{012}: $\bar{\alpha}_1 = \bar{\alpha}_2 = 0$,
$\bar{\alpha}_3 = \bar{\alpha}_4 = |\mu|$



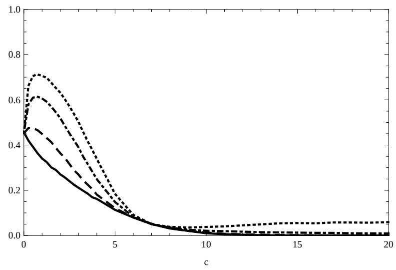
(i) S_{012}:
$\bar{\alpha}_1 = \bar{\alpha}_2 = \bar{\alpha}_3 = \bar{\alpha}_4 = |\mu|$

Figure 1. Asymptotic size, fixed initial condition

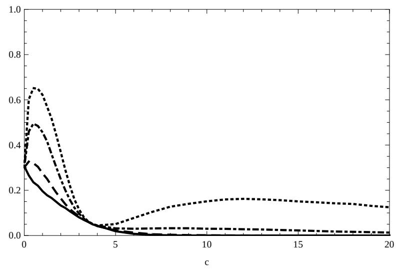
$\mu = 0$: ——— , $\mu = 2$: — — , $\mu = 4$: — · —, $\mu = 6$: · · ·



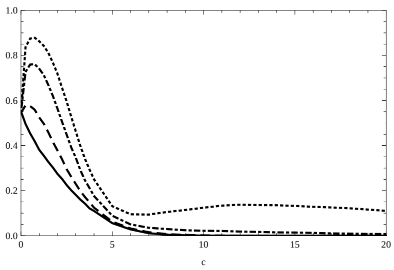
(a) S_0



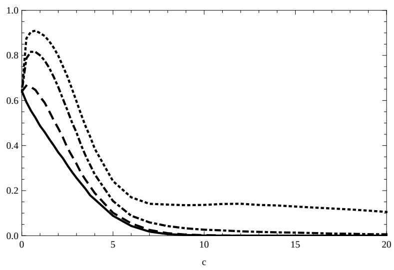
(b) S_2



(c) S_1



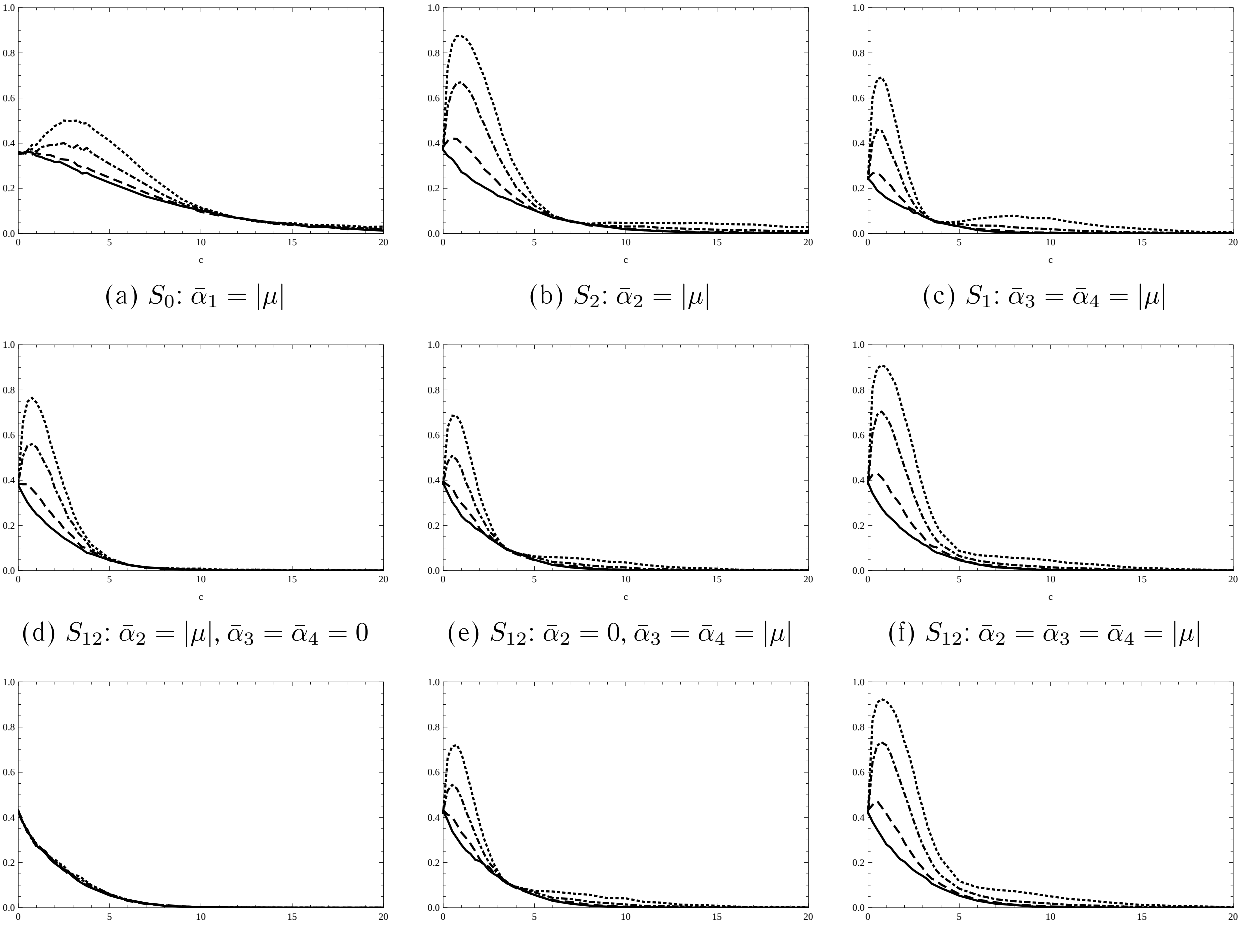
(d) S_{12}



(e) S_{012}

Figure 2. Asymptotic size, random initial condition, $\alpha_i \sim N(0, \sigma_\alpha^2)$, $i = 1, 2, 3, 4$

$\mu = 0 :$ —— , $\mu = 2 :$ — — , $\mu = 4 :$ — · —, $\mu = 6 :$ · · ·



(a) S_0: $\bar{\alpha}_1 = |\mu|$

(b) S_2: $\bar{\alpha}_2 = |\mu|$

(c) S_1: $\bar{\alpha}_3 = \bar{\alpha}_4 = |\mu|$

(d) S_{12}: $\bar{\alpha}_2 = |\mu|, \bar{\alpha}_3 = \bar{\alpha}_4 = 0$

(e) S_{12}: $\bar{\alpha}_2 = 0, \bar{\alpha}_3 = \bar{\alpha}_4 = |\mu|$

(f) S_{12}: $\bar{\alpha}_2 = \bar{\alpha}_3 = \bar{\alpha}_4 = |\mu|$

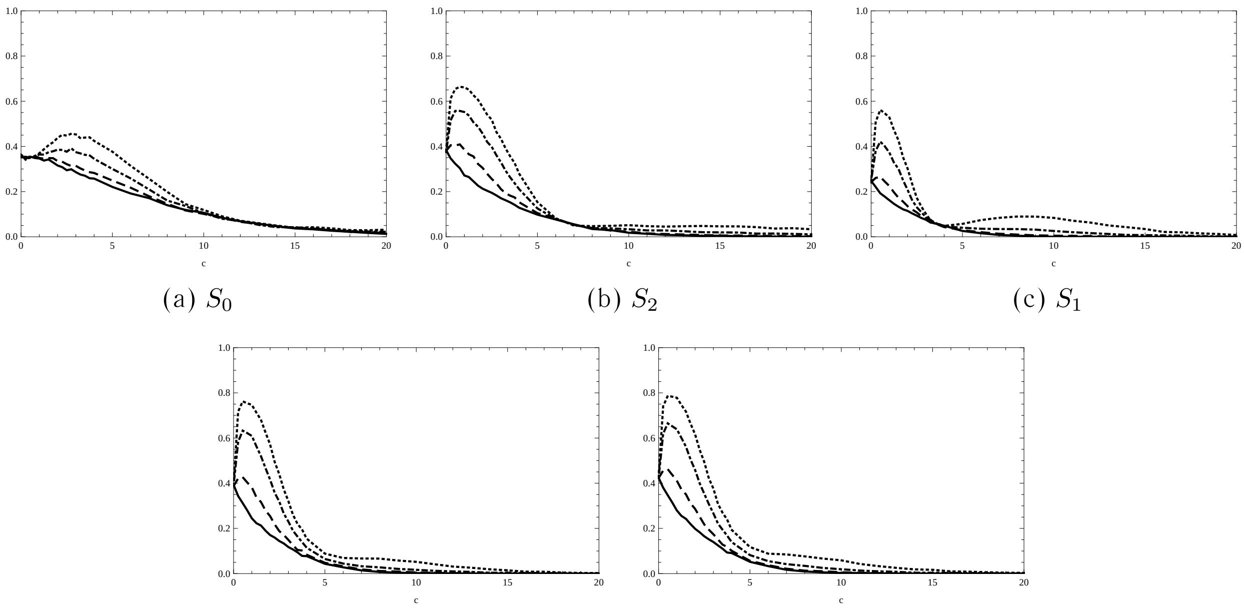
(g) S_{012}: $\bar{\alpha}_1 = |\mu|$,
$\bar{\alpha}_2 = \bar{\alpha}_3 = \bar{\alpha}_4 = 0$

(h) S_{012}: $\bar{\alpha}_1 = \bar{\alpha}_2 = 0$,
$\bar{\alpha}_3 = \bar{\alpha}_4 = |\mu|$

(i) S_{012}:
$\bar{\alpha}_1 = \bar{\alpha}_2 = \bar{\alpha}_3 = \bar{\alpha}_4 = |\mu|$

Figure 3. Finite sample size, fixed initial condition

$\mu = 0$: —— , $\mu = 2$: — — , $\mu = 4$: — · —, $\mu = 6$: · · ·



(a) S_0 (b) S_2 (c) S_1

(d) S_{12} (e) S_{012}

Figure 4. Finite sample size, random initial condition, $\alpha_i \sim N(0, \sigma_\alpha^2)$, $i = 1, 2, 3, 4$

$\mu = 0 :$ —— , $\mu = 2 :$ — — , $\mu = 4 :$ — · —, $\mu = 6 : \cdot \ \cdot \ \cdot$