On GLS-detrending for deterministic seasonality testing*

Anton Skrobotov†
The Russian Presidential Academy of National Economy and Public Administration

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Abstract

In this paper we propose tests based on GLS-detrending for testing the null hypothesis of deterministic seasonality. Unlike existing tests for deterministic seasonality, our tests do not suffer from asymptotic size distortions under near integration. We also investigate the behavior of the proposed tests when the initial condition is not asymptotically negligible.
Key words: Stationarity tests, KPSS test, seasonality, seasonal unit roots, deterministic seasonality, size distortion, GLS-detrending.
JEL: C12, C22.

1 Introduction

Deterministic seasonality describes the behavior of a time series in which the unconditional means change in different seasons of the year. One way to record this is a seasonal dummies representation. On the other hand, the presence of a seasonal unit root in the data could distort the seasonal correction procedure for the time series, so dividing the processes using deterministic seasonality and seasonal unit root processes is important in the time series analysis. Most of the work, following Hylleberg *et al.* (1990) (hereinafter HEGY), focuses on testing for seasonal unit roots at different frequencies against the alternative that all the roots are less than one. This procedure is related to the testing of the unit root (at zero frequency) against the alternative of stationarity in the nonseasonal case.

Similar to the nonseasonal case (the stationarity test against the alternative hypothesis of a unit root, see Kwiatkowski *et al.* (1992)) the procedures for deterministic seasonality testing against the case of the presence of at least one seasonal unit root has also been developed. The problem of deterministic seasonality testing was first considered by Canova and Hansen (1995).

*We thank Editor, Associate Editor and two anonymous referees for helpful suggestions.
†Adress correspondence to: Institute of Applied Economic Research, Russian Presidential Academy of National Economy and Public Administration, 82, Vernadsky pr., 117571, Moscow, Russia.
E-mail: antonskrobotov@gmail.com

Taylor (2003a) analyzed a more general formulation in the context of the construction of locally mean most powerful invariant (LMMPI) tests. Taylor (2003b) also shows that under the presence of unattended unit roots, i.e. if the null hypothesis is tested against the alternative of a specific number of unit roots, but actually the series contains a number of additional roots at other frequencies, then the test statistic under the null will be $O_p((TS_T)^{-1})$. This means the test will be conservative (and size distortion will increase as the sample size increases). To solve this problem, the author proposed the use of pre-filtered data similar to HEGY test. Kurozumi (2002) investigated the limiting properties of the Canova and Hansen test. The author has found the limiting distributions of tests by using the Fredholm approach (see Tanaka (1996)), and also has shown that the power of the tests depends not only on the local parameter, c, but also on the reciprocal of the spectral density of the stationary component of the time series at frequency π or $\pi/2$.

There are alternative approaches to testing for deterministic seasonality. In Caner (1998), in contrast to the non-parametric approach proposed by Canova and Hansen (1995) and Taylor (2003a), the parametric autocorrelation correction of errors according to the Leybourne and McCabe (1994) principle is used (but under stronger assumptions). More precisely, Caner (1998) uses the residuals from regression not only on the deterministic component, but also on a sufficient number of lagged dependent variables. In this case, there is no need to construct a nonparametric estimator of long-run variance and a conventional variance estimate of pre-whitened data should be used. The distributions of the test statistics coincide with the results obtained in Canova and Hansen (1995) and Taylor (2003a). Another approach for deterministic seasonality testing was developed by Tam and Reinsel (1997, 1998) in the context of testing for the unit root in the MA component of a time series. For comparisons of this approach with the Canova and Hansen tests, see Ghysels and Osborn (2001, Sections 2.4.3, 2.4.4).

One problem is that under near integration all seasonal unit root tests at different frequencies will have an asymptotic size equal to unity (see Müller (2005) for the nonseasonal case). One solution to this problem (in the nonseasonal case) has been proposed by Harris *et al.* (2007) (henceforth HLM), where the authors used a (quasi) GLS-detrending to construct the test statistics. Here we generalize their approach to a seasonal case and we test the hypothesis of deterministic seasonality (local to a seasonal unit root) against the alternative of a seasonal unit root.

This paper is organized as follows. Section 2 describes the data generating process (DGP) and assumptions about errors and initial conditions (the assumptions about the initial condition follow from Harvey *et al.* (2008) (henceforth HLT)). In Section 3 we propose the procedure of seasonal GLS-detrending for stationarity test statistics. The Monte-Carlo simulation results (asymptotic and finite sample) are given in Section 4. The results are formulated in the Conclusion.

2 Model

Consider quarterly DGP such that

$$y_{4n+s} = \mu_{4n+s} + u_{4n+s},\ s = -3, \dots, 0,\ n = 1, \dots, N, \quad (1)$$

$$a(L)u_{4n+s} = \varepsilon_{4n+s},\ s = -3, \dots, 0,\ n = 2, \dots, N, \quad (2)$$

where $a(L) = 1 - \sum_{j=1}^{4} a_j L^j$ is a fourth order autoregressive polynomial, L is the lag operator such that $L^{4j+k} y_{4n+s} = y_{4(n-j)+s-k}$, $T = 4N$ is the number of observations (N is the span in years of the sample data). The errors ε_{4n+s} are assumed to be a zero mean process, the long-run variance of which is bounded and strongly positive at zero and seasonal spectral frequencies, $\omega_k = 2\pi k/4$, $k = 0, 1, 2$.

The deterministic component $\mu_{4n+s} = \mu_t$ is defined as a linear combination of spectral indicator variables, corresponding to the zero and seasonal frequencies, $z_{t,0} = 1$, $z_{t,1} = (\cos[2\pi t/4], \sin[2\pi t/4])'$ and $z_{t,2} = (-1)^t$. Define the vector $Z_t = (z_{t,0}, z'_{t,1}, z_{t,2})'$ and the deterministic component $\mu_t = d'_\xi Z_{t,\xi}$ for three possible cases, $\xi = 1, \ldots, 3$ (see Smith and Taylor (1998)). The first case corresponds to the constants at zero and seasonal frequencies, $Z_{t,1} = Z_t$, the second case also allows for the trend at zero frequency, $Z_{t,2} = (Z'_t, z_{t,0}t)'$, the third case allows for the trend at zero and seasonal frequencies, $Z_{t,3} = (Z'_t, tZ'_t)'$.

The polynomial $a(L)$ can be factorized as $\prod_{k=0}^{2} \omega_k(L) = (1-a_0L)(1-2\beta_1 L+(a_1^2+\beta_1^2)L^2)(1-a_2L)$. We are interested in testing for deterministic seasonality (local to the seasonal unit root), in other words to test the hypothesis $H_0 : c_i \geq \bar{c}_i > 0$ for all i in $a_i = 1 - c_i/T$, against the alternative about the unit root at least one of the frequencies, i.e. $H_1 : c_i = 0$ for at least one i, where $\bar{c}_i$ is the minimal amount of mean reversion for the specific frequency under the null hypothesis. The null hypothesis H_0 can be partitioned as $H_0 = \bigcap_{k=0}^{2} H_{0,c_k}$, where $H_{0,c_i} : a_i = 1 - c_i/T$, $i = 0, 2$ and $H_{0,c_1} : a_1 = 1 - c_1/T$, $\beta_1 = 0$. In other words, testing the null hypothesis of stationarity (local to the unit root) at zero frequency, $\omega_0 = 0$, against the alternative of a unit root at this frequency is equivalent to testing $H_{0,c_0} : c_0 \geq \bar{c}_0 > 0$ against $H_{0,c_0} : c_1 = 0$. Similarly, testing the null hypothesis for stationarity (local to unit root) at the Nyquist frequency, $\omega_2 = \pi$, against the alternative of a unit root at this frequency is equivalent to testing $H_{0,c_2} : c_2 \geq \bar{c}_2 > 0$ against $H_{2,c_2} : c_2 = 0$, and testing the null hypothesis of stationarity (local to unit root) at seasonal harmonic frequencies, $(\pi/2, 3\pi/2)$, against the alternative of unit roots at these frequencies is equivalent to testing $H_{0,c_1} : c_1 \geq \bar{c}_1 > 0$ against $H_{1,c_1} : c_1 = 0$.

This approach differs to the usual testing for deterministic seasonality, where either the local asymptotic behavior is not considered or parameters related to the signal-to-noise ratio are assumed to be local (see Taylor (2003*a*)). The reason for considering near integration is the same as in Müller (2005): it explains the increasing size in finite samples for highly autocorrelated stationary series. As demonstrated in Müller (2005) in the context of nonseasonal models, the conventional KPSS test with the bandwidth parameter in the long-run variance estimator increasing at a slower rate than the length of the sample leads to an asymptotic size equal to unity under the null hypothesis of near integration. It is easy to show that the same problem arises in the seasonal models, if we use the tests of Canova and Hansen (1995), Taylor (2003*a*) and Taylor (2003*b*), *inter alia*. One way to solve this problem will be considered in the next section where we extend the HLM test to a seasonal case.

Also we set the initial condition u_i, $i = 1, \ldots, 4$, according to Assumption 1 (see HLT).

Assumption 1 *Under H_0 with $c < 0$, the initial conditions are generated according to*

$$u_i = \alpha_i \sqrt{\omega_\varepsilon^2/(1-\rho_N^2)},\ i = 1, \ldots, 4, \tag{3}$$

where $\rho_N = 1 - c/N$ and $\alpha_i \sim IN(\mu_{\alpha,i}\mathbb{I}(\sigma_\alpha^2 = 0), \sigma_\alpha^2)$, $i = 1, \ldots, 4$, independent of ε_{4n+s}, $s = -3, \ldots, 0$, $n = 2, \ldots, N$. For $c = 0$, under H_1, the initial conditions can be set equal to zero, $u_i = 0$, $i = 1, \ldots, 4$, without loss of generality, due to the exact similarity of the tests to the initial conditions in this case.

In this assumption α_i controls the magnitude of the initial condition in season i relative to the magnitude of the standard deviation of a stationary seasonal AR(1) process with parameter ρ_N

and innovation long-run variance ω_ε^2. The form given for the u_i allow the initial conditions to be either random and of $O_p(N^{1/2})$, or fixed and of $O(N^{1/2})$ depending on the value of variance σ_α^2 (> 0 or 0, respectively).

3 Deterministic seasonality testing based on GLS-detrending

HLM proposed the following test using a (quasi) GLS-detrended series. More precisely, let $\tilde{u}_t^\xi$ be the residuals from regression $y_{\bar{c}} = y_t - \bar{\rho}_T y_{t-1}$ on $Z_{i,\bar{c}} = z_t - \bar{\rho}_T z_{t-1}$, $t = 2, \ldots, T$, where $z_t = 1$ in constant case ($\xi = \mu$) and $z_t = (1, t)'$ in trend case ($\xi = \tau$) and $\bar{\rho}_T = 1 - \bar{c}/T$. Then the $S^\xi(\bar{c})$ test is constructed as following:

$$S^\xi(\bar{c}) = \frac{T^{-2} \sum_{t=2}^{T} (\sum_{j=2}^{t} \tilde{u}_j^i)^2}{\hat{\omega}^2}, \tag{4}$$

where the kernel based long-run variance estimator $\hat{\omega}^2$ is calculated by using GLS-detrended residuals $\tilde{u}_t^i$.

For a seasonal time series consider the following GLS-transformation (see also Rodrigues and Taylor (2007) in the context of seasonal unit root testing), by using a vector $\mathbf{c} = (\bar{c}_0, \bar{c}_1, \bar{c}_2)$. Let the series y_c and $\mathbf{Z}_{\xi,\mathbf{c}}$ be defined as

$$\begin{aligned} y_{\mathbf{c}} &= (\Delta_{\mathbf{c}} y_{S+1}, \ldots, \Delta_{\mathbf{c}} y_T)' \\ \mathbf{Z}_{\xi,\mathbf{c}} &= (\Delta_{\mathbf{c}} Z_{S+1,\xi}, \ldots, \Delta_{\mathbf{c}} Z_{T,\xi})', \end{aligned}$$

where

$$\Delta_{\mathbf{c}} = 1 - \sum_{j=1}^{S} a_j^{\mathbf{c}} L^j = \left(1 - \left(1 - \frac{\bar{c}_0}{T}\right) L\right) \left(1 + \left(1 - \frac{\bar{c}_2}{T}\right) L\right) \left(1 + \left(1 - \frac{\bar{c}_1}{T}\right)^2 L^2\right)$$

(respectively, $\Delta_{\mathbf{0}} = 1 - L^S$). GLS-detrended series are the OLS-residuals from regression y_c on $\mathbf{Z}_{\xi,\mathbf{c}}$. Define these residuals as $\tilde{u}_{t,\xi}$.

Before constructing the test statistics, we note that one of the basic principles of unit root testing in a seasonal time series is that when we test for a unit root at a specific frequency, the data should be prefiltered to reduce the order of integration at each of the remaining (unattended) frequencies by one (see HEGY and Taylor (2003*b*)). If we are testing the stationarity against the presence of a unit root at zero frequency it is necessary to use the data $y_{0,t} = (1 + L)(1 + L^2)y_t$ instead of y_t, and then perform GLS-detrending. Accordingly for testing the stationarity at Nyquist frequency it is necessary to use the data $y_{2,t} = (1 - L)(1 + L^2)y_t$, and for testing the stationarity at the seasonal harmonic frequencies it is necessarily to use the data $y_{1,t} = (1 - L)(1 + L)y_t$. However, due to the principle of constructing the GLS-detrended series (see Appendix) there is no need to prefilter unit roots at specific frequencies. Moreover, based on preliminary simulations we conclude that the use of prefiltered data reduces the finite sample power considerably while even for $T = 100$ (and i.i.d. errors) our GLS-based test with no prefiltering of unit roots has size curves very close to asymptotic.

Thus, a final test can be written as follows:

$$S_{\xi,j}(\mathbf{c}) = T^{-2} tr \left[(C_j' \hat{\Omega}_Z C_j)^{-1} C_j' \sum_{t=1}^{T} \left(\sum_{s=1}^{t} \tilde{u}_{s,\xi}^j Z_s \right) \left(\sum_{s=1}^{t} \tilde{u}_{s,\xi}^j Z_s' \right) C_j \right], \tag{5}$$

where $\hat{\Omega}_Z$ is a consistent long-run variance estimator of $Z_t\varepsilon_t$, where

$$\hat{\Omega}_Z = \sum_{l=-T+1}^{T-1} k\left(\frac{j}{S_T}\right)\hat{\Gamma}(l), \tag{6}$$

with the bandwidth parameter $S_T \to \infty$, $S_T/T^{1/2} \to 0$ and the autocovariance estimator

$$\hat{\Gamma}(l) = T^{-1}\sum_{s=l+1}^{T} \tilde{u}^j_{s,\xi} Z_t \tilde{u}^j_{s-l,\xi} Z'_{s-l},\ \hat{\Gamma}(-l) = \hat{\Gamma}(l),\ l \geq 0. \tag{7}$$

The matrix C_j depends on the frequencies at which we test the stationarity. For zero frequency C_0 is the first column of identity matrix $\mathbf{I}_4$, for Nyquist frequency C_2 is the fourth column of $\mathbf{I}_4$, for seasonal harmonic frequencies C_1 is the matrix consisting of the second and third columns of $\mathbf{I}_4$. For testing the stationarity at all seasonal frequencies the C_{12} matrix contains the second, third and forth columns, and for testing the stationarity at all seasonal and nonseasonal frequencies C_{012} is the identity matrix $\mathbf{I}_4$.

The following proposition gives the limiting distributions of all test statistics to test the stationarity (local to the seasonal unit root) at specific frequencies under the null and alternative hypotheses.

Proposition 1 *Let $\{y_{Sn+s}\}$ be generated as (1)-(2) under Assumption 1. For $i = 1, \ldots, 4$ we define*

$$K_{ic}(r) = \begin{cases} W_i(r), & \text{if } c = 0, \\ \bar{\alpha}_i(e^{rc} - 1)(-2c)^{-1/2} + W_{ic}(r), & \text{if } c < 0, \end{cases}$$

where $W_i(r)$, $i = 1, \ldots, 4$, are independent standard Wiener processes, $W_{ic}(r)$, $i = 1, \ldots, 4$, are independent Ornstein-Uhlenbeck processes, and the spectral magnitudes $\bar{\alpha}_i$ are defined as

$$\begin{aligned} \bar{\alpha}_1 &= (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)/2, \\ \bar{\alpha}_2 &= (-\alpha_1 + \alpha_2 - \alpha_3 + \alpha_4)/2, \\ \bar{\alpha}_3 &= (\alpha_4 - \alpha_2)/\sqrt{2}, \\ \bar{\alpha}_4 &= (\alpha_3 - \alpha_1)/\sqrt{2}. \end{aligned}$$

Then

$$S_j(\bar{c}_j) \Rightarrow \int_0^1 H_{jc,\bar{c}_j}(r)^2 dr,\ j = 0, 2, \tag{8}$$

$$S_1(\bar{c}_1) \Rightarrow \int_0^1 (H_{1c,\bar{c}_1}(r) + H_{3c,\bar{c}_1}(r))^2 dr, \tag{9}$$

$$S_{12}(\bar{c}_1) \Rightarrow \int_0^1 (H_{1c,\bar{c}_1}(r) + H_{2c,\bar{c}_2}(r) + H_{3c,\bar{c}_1}(r))^2 dr, \tag{10}$$

$$S_{012}(\bar{c}_1) \Rightarrow \int_0^1 (H_{0c,\bar{c}_0}(r) + H_{1c,\bar{c}_1}(r) + H_{2c,\bar{c}_2}(r) + H_{3c,\bar{c}_1}(r))^2 dr, \tag{11}$$

where

$$H_{ic,\bar{c}_j}(r) = K_{ic}(r) + \bar{c}_j \int_0^r K_{ic}(s)ds - r\left[K_{ic}(1) + \bar{c}_j \int_0^1 K_{ic}(s)ds\right],$$

$i = 1, \ldots, 4$, *for* $j = 0$, $\xi = 0$ *and for* $j = 1, 2$, $\xi = 0, 1$, *and*

$$H_{ic,\bar{c}_j}(r) = H_{ic,\bar{c}_j}(r) - 6r(1-r)\int_0^1 H_{ic,\bar{c}_j}(s)ds,$$

$i = 1, \ldots, 4$, *for* $j = 0$, $\xi = 1, 2$ *and for* $j = 1, 2$, $\xi = 3$, *and* $\Rightarrow$ *denotes weak convergence.*

The proof of this proposition follows from HLM, Taylor (2003a) and HLT and is omitted for brevity. Similar to HLM, it can be shown that for $c_j = \bar{c}_j$ the limiting distribution of the S_j test coincides with the results obtained in Canova and Hansen (1995) and Taylor (2003a) (under the null hypothesis) and does not depend on initial conditions. Similar results will be occurred for $c_1 = \bar{c}_1$ and $c_2 = \bar{c}_2$ for the S_{12} test and for $c_0 = \bar{c}_0$, $c_1 = \bar{c}_1$ and $c_2 = \bar{c}_2$ for the S_{012} test. See Section 4.2 for a discussion of calculating the finite sample critical values.

Also it can be noted that the limiting distributions depend not on the values of the initial conditions α_i, but on the so-called spectral initial conditions $\bar{\alpha}_i$ in the terminology of HLT. Hence for some given nonzero initial conditions, their linear combination may be zero and some tests will not depend on their magnitudes.

4 Monte-Carlo simulations

4.1 Asymptotic results

In this subsection, we analyze the asymptotic behavior of S_0, S_2, S_1, S_{12} and S_{012} tests under various magnitudes for the initial conditions[1]. Figure 1 shows results for the case of a fixed initial condition ($\sigma_\alpha^2 = 0$ in Assumption 1), while Figure 2 shows results for the case of a random initial condition ($\sigma_\alpha^2 > 0$ in Assumption 1). Everywhere we consider a model with seasonal constants and a nonseasonal trend (since this model is most often used in practice) and use $\bar{c}_0 = 13.5$, $\bar{c}_2 = 7$, $\bar{c}_1 = 3.75$.

Parts (a), (b) and (c) in Figure 1 represent, respectively, asymptotic size ($a(L) = 1 - (1 - c/N)L^4$, $c > 0$) and power ($c = 0$) of S_0, S_2 and S_1 tests for various magnitudes of initial conditions $\bar{\alpha}_1$, $\bar{\alpha}_2$ and $\bar{\alpha}_3 = \bar{\alpha}_4$, respectively. The magnitudes of the initial conditions $\bar{\alpha}_i = |\mu| = \{0, 2, 4, 6\}$, $i = 1, \ldots, 4$ are used for all the tests being considered. Parts (d), (e) and (f) represent results for the S_{12} test under: $\bar{\alpha}_2 = |\mu|$, $\bar{\alpha}_3 = \bar{\alpha}_4 = 0$; $\bar{\alpha}_2 = 0$, $\bar{\alpha}_3 = \bar{\alpha}_4 = |\mu|$; $\bar{\alpha}_2 = \bar{\alpha}_3 = \bar{\alpha}_4 = |\mu|$, respectively. Parts (g), (h) and (i) represent results for the S_{012} test under: $\bar{\alpha}_1 = |\mu|$, $\bar{\alpha}_2 = \bar{\alpha}_3 = \bar{\alpha}_4 = 0$; $\bar{\alpha}_1 = \bar{\alpha}_2 = 0$, $\bar{\alpha}_3 = \bar{\alpha}_4 = |\mu|$; $\bar{\alpha}_1 = \bar{\alpha}_2 = \bar{\alpha}_3 = \bar{\alpha}_4 = |\mu|$, respectively. Corresponding results for the case of random initial conditions are given in Figure 2 for $\sigma_\alpha^2 = |\mu| = \{0, 2, 4, 6\}$, $i = 1, \ldots, 4$. In all cases critical values are obtained at $c_j = \bar{c}_j$.

The size curves of S_0, S_2 and S_1 tests are tangent to each other for different initial conditions at points $c_j = \bar{c}_j$, which confirms the invariance with respect to the initial conditions of each of

[1]Here and in the following subsection results are obtained by simulations of the limiting distributions in Proposition 1, approximating the Wiener process using $i.i.d.N(0, 1)$ random variates and with integrals approximated by normalized sums of 1,000 steps, with 50,000 replications.

the tests at $c = \bar{c}_j$. When $c < \bar{c}_j$ the size of each test grows as the corresponding spectral initial condition is increased, although the size distortion is less pronounced for the S_0 test (intuitively because a larger value is used for $\bar{c}_j$). For $c > \bar{c}_j$ the size of the test is lower that the nominal one, except for the S_1 test with $|\mu| = 6$ (because a very small value of $\bar{c}_j$ is used). Similar behavior is observed for S_{12} and S_{012}.

For the case of random initial conditions, the size distortions for $c < \bar{c}_j$ are smaller that in case of fixed initial conditions, but larger than for $c > \bar{c}_j$.

4.2 Finite sample comparisons

In finite samples, when the error term ε_t is i.i.d., the size curves are close to asymptotic. However, if ε_t has a MA form, the power can be very low. One way to solve this issue is to sacrifice the size at some small $c = \bar{c}_j$ (proposed above) and use a higher value of $\bar{c}_j$, which will control the size at a higher c. One alternative is to use parameters $\bar{c}_j$ such that the asymptotic size of S_0, S_2 and S_1 tests (e.g., in case of a seasonal constant and nonseasonal trend) is equal to 0.5 at $c_0 = 13.5$, $c_2 = 7$, $c_1 = 3.75$. These new values of $\bar{c}_j$ are equal to $\bar{c}_0 = 29.55$, $\bar{c}_2 = 20.05$ and $\bar{c}_1 = 10.7$ (in case of seasonal trend $\bar{c}_1 = 17.75$ and $\bar{c}_2 = 29.55$). Corresponding values of power in finite samples ($T = 300$) for S_0, S_2, S_1, S_{01} and S_{012} tests will then be equal to 0.90, 0.86, 0.85, 0.97 and 0.99, respectively. Of course, the researcher can choose any values of $\bar{c}_j$ for his desired trade-off between size and power. However, if we choose the greater value of $\bar{c}_j$, the critical values are more downward-biased. Therefore, for a hypothesis testing the finite sample, critical values should be used. Moreover, for large $\bar{c}_j$ the size distortion increases under AR or MA error.[2]

In this subsection we investigate the finite sample behavior of tests based on DGP (1)-(2) with $\xi = 2$ (seasonal intercept, nonseasonal trend) and Assumption 1. Results are presented for $T = 300$ with 10,000 replications and, without loss of generality, with the absence of the deterministic term. For simplicity, the initial conditions are assumed to be random. The long-run variance estimator (6) is constructed by using a quadratic spectral window and automatic bandwidth selection based on AR(1) approximation (see Andrews (1991)). As a comparison, we use conventional deterministic seasonality tests proposed by Taylor (2003*b*), where the long-run variance estimator is constructed by using a Bartlett kernel with bandwidth $S_T = 6$.[3] Because these two tests have different natures, we compare their size ($c > 0$) by fixing power ($c = 0$) at a predetermined level. More precisely, critical values are obtained so that the power is equal to 0.70 for $T = 100$. Based on obtained critical values we calculate the size of all tests for $c \in \{5, 10, \ldots, 30\}$. It should be noted that the choice of setting power at a value of 0.70 is not crucial, and this value is chosen for the convenience of visualizing the comparison. In simulations we use the values of $\bar{c}_0 = 29.55$, $\bar{c}_2 = 20.05$ and $\bar{c}_1 = 10.7$. Other values of $\bar{c}_j$ produce similar results.

Tables 1-9 show the power-adjusted size of a conventional deterministic seasonality test (denoted as S_j) and our proposed GLS-based test (denoted as $S_j(\mathbf{c})$) under various DGP, including i.i.d, AR and MA forms of error. More precisely, $a(L) = 1 - (1 - c/N)L^4$,[4] and the error ε_t have the both AR form,

$$(1 - \Phi L^4)\varepsilon_t = e_t, \tag{12}$$

[2]Ox-code for computing the finite sample critical values based on the parameters $\bar{c}_j$ and the corresponding power of all tests are available in https://sites.google.com/site/antonskrobotov/.

[3]This choice in a long-run variance estimation shows the best properties in finite samples.

[4]Results in which unit roots are present at some but not all frequencies are similar.

or MA form,

$$\varepsilon_t = (1 - \Theta L^4)e_t, \tag{13}$$

where $e_t \sim i.i.d.N(0,1)$. We perform simulations with $\Phi = \pm 0.3, \pm 0.5$ for (12) and with $\Theta = \pm 0.3, \pm 0.5$ for (13), and also for $\varepsilon_t = e_t$.

For i.i.d. errors in most cases the $S_j(\mathbf{c})$ tests have a smaller size than the S_j, except for a small number of cases for small c. The results are similar for AR errors except for a small initial conditions for the S_{12} and S_{012} tests. For large Φ, the S_{12} and S_{012} tests outperform the $S_{12}(\mathbf{c})$ and $S_{012}(\mathbf{c})$ even for large initial conditions. For MA errors in most DGPs the GLS-based tests outperform conventional tests although for negative Θ the size of $S_1(\mathbf{c})$ is much higher than the one of S_1. For positive Θ, the results are similar to AR cases. In general, our proposed GLS-based tests show the best properties in view of the fact that they have better size control specified for the local parameters c.

It should be noted that GLS-based tests are superior to conventional tests for deterministic seasonality only in finite samples, as the latter are consistent. Due to the nature of their construction, GLS-tests are not consistent in the sense that their power will not tend to unity with the growth of the sample size. Therefore, in large samples, conventional tests will outperform GLS-based tests.

5 Conclusion

In this paper, we consider GLS-detrendng in the context of deterministic seasonality testing (local to a seasonal unit root) against the alternative of a seasonal unit root. Unlike existing tests, the proposed tests do not suffer from asymptotic size distortions in the local to unity autoregression parameters and, at the same time, have a well-controlled size for a given level of (seasonal) mean reversion. The behavior of the proposed tests was also analyzed for fixed and random initial conditions of various magnitudes. Finite sample behavior confirms the superiority of GLS-based tests over conventional tests for deterministic seasonality.

Appendix

Proof of Proposition 1: Without loss of generality let the deterministic term $\mu_t = 0$. Also for convenience define $x_{4n+s} = u_{4n+s} - u_s$. For simplicity consider the case with a seasonal intercept and no trend, $\xi = 1$. The extension on nonseasonal or seasonal trends are similar. GLS-residuals $\tilde{u}_{t,1}$ are defined as

$$\tilde{u}_{t,1} = \Delta_{\mathbf{c}} x_t - \hat{\mu}_t,$$

where $\hat{\mu}_t = \hat{d}_1' Z_{t,1}$, $\hat{d}_1'$ is a vector of GLS-estimates for spectral indicator variables $Z_{t,1}$. Note that now $\Delta_{\mathbf{c}} y_t$ can be decomposed as

$$\Delta_{\mathbf{c}} x_t = \left((1 - L^4) + \frac{\bar{c}_0}{T}(L + L^2 + L^3 + L^4) + \frac{\bar{c}_2}{T}(-L + L^2 - L^3 + L^4) + \frac{2\bar{c}_1}{T}(-L^2 + L^4) \right) x_t + o_p(1)$$

Consider deterministic seasonality testing against a unit root at the zero frequency ($\omega_0 = 0$).

Then

$$\begin{aligned}
T^{-1/2}\sum_{i=5}^{t}\tilde{u}_{t,1} &= T^{-1/2}\sum_{i=5}^{t}\Delta_{\mathbf{c}}x_t - T^{-1/2}\sum_{i=5}^{T}\hat{\mu}_t \\
&= T^{-1/2}\sum_{i=5}^{t}\Delta_{\mathbf{c}}x_t - tT^{-3/2}\sum_{i=5}^{T}\Delta_{\mathbf{c}}x_t \\
&= T^{-1/2}(x_t + x_{t-1} + x_{t-2} + x_{t-3} - x_5 - x_6 - x_7 - x_8) \\
&\quad +\bar{c}_0 T^{-3/2}\sum_{i=5}^{t}(x_{i-1} + x_{i-2} + x_{i-3} + x_{i-4}) \\
&\quad -t[T^{-1/2}(x_T + x_{T-1} + x_{T-2} + x_{T-3} - x_5 - x_6 - x_7 - x_8) \\
&\quad -\bar{c}_0 T^{-3/2}\sum_{i=5}^{T}(x_{t-1} + x_{t-2} + x_{t-3} + x_{t-4})] + o_p(1).
\end{aligned}$$

The second equality follows from the orthogonality of spectral indicator variables $Z_{t,1}$, and the third equality from $x_1 = \cdots = x_4 = 0$ and from

$$\bar{c}_2 T^{-3/2}\sum_{i=5}^{t}(-x_{t-1} + x_{t-2} - x_{t-3} + x_{t-4}) = O_p(T^{-1})$$

and

$$2\bar{c}_1 T^{-3/2}\sum_{i=5}^{t}(-x_{t-2} + x_{t-4}) = O_p(T^{-1}).$$

Thus,

$$T^{-1/2}\sum_{i=5}^{[rT]}\tilde{u}_{t,1} \Rightarrow \omega_{1,Z} H_{ic,\bar{c}_0}(r),$$

where $\omega_{1,Z}^2$ is a (1,1) element of covariance matrix Ω_Z. Further, applying the CMT, we obtain

$$T^{-1}\sum_{t=5}^{T}(T^{-1/2}\sum_{i=5}^{t}\tilde{u}_{t,1})^2 \Rightarrow \omega_{1,Z}^2\int_0^1 H_{ic,\bar{c}_0}(r)^2.$$

The consistency of $\hat{\omega}_{1,Z}^2$ is proved similarly HLM, p. 362.

For analysis of the limiting distribution of the test statistics at Nyquist frequency ($\omega_2 = \pi$) and at harmonic seasonal frequencies ($\omega_1 = \pm\pi/2$), consider decomposition of the polynomial $\Delta_{\mathbf{c}}$ up to the order T^{-1}:

$$(1-L)(1+L+L^2+L^3) + \frac{\bar{c}_0}{T}(L+L^2+L^3+L^4) + \frac{\bar{c}_2}{T}(-L+L^2-L^3+L^4) + \frac{2\bar{c}_1}{T}(-L^2+L^4). \quad \text{(A1)}$$

When we obtained the limiting distribution at zero frequency, the third and fourth terms were asymptotically negligible. If we analyze the behavior of the test statistics at Nyquist frequency, then instead of considering partial sums $T^{-1/2}\sum_{i=5}^{t}\tilde{u}_{t,1}$, we should consider partial sums $T^{-1/2}\sum_{i=5}^{t}\tilde{u}_{t,1}(-1)^i$. This leads to the fact that the first term in (A1) becomes $(1-L)(-1+L-$

$L^2 + L^3$), and the second and fourth terms can be neglected, so that only the modified first and third terms are remain. There is a similar case with testing at harmonic seasonal frequencies, when the second and third terms in (A1) can be neglected.

Note also that the specified type of polynomial (A1) presupposes taking the filtered data, so additional filtering for possibly unattended unit roots is not required.

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Table 1. Power adjusted size of tests, $i.i.d$ errors

c	S_0	$S_0(\mathbf{c})$	S_2	$S_2(\mathbf{c})$	S_1	$S_1(\mathbf{c})$	S_{12}	$S_{12}(\mathbf{c})$	S_{012}	$S_{012}(\mathbf{c})$
					$\mu = 0$					
5	0.562	0.599	0.400	0.384	0.187	0.222	0.090	0.155	0.082	0.155
10	0.367	0.357	0.236	0.153	0.044	0.023	0.009	0.013	0.006	0.012
15	0.229	0.154	0.139	0.038	0.009	0.001	0.001	0.000	0.000	0.000
20	0.138	0.048	0.082	0.006	0.002	0.000	0.000	0.000	0.000	0.000
25	0.087	0.010	0.051	0.001	0.000	0.000	0.000	0.000	0.000	0.000
30	0.054	0.001	0.031	0.000	0.000	0.000	0.000	0.000	0.000	0.000
					$\mu = 2$					
5	0.622	0.642	0.523	0.471	0.376	0.264	0.305	0.239	0.301	0.243
10	0.430	0.384	0.295	0.170	0.086	0.024	0.033	0.016	0.027	0.014
15	0.267	0.162	0.164	0.039	0.017	0.001	0.002	0.001	0.001	0.000
20	0.160	0.048	0.095	0.006	0.003	0.000	0.000	0.000	0.000	0.000
25	0.096	0.009	0.056	0.001	0.001	0.000	0.000	0.000	0.000	0.000
30	0.061	0.001	0.034	0.000	0.000	0.000	0.000	0.000	0.000	0.000
					$\mu = 4$					
5	0.709	0.702	0.667	0.591	0.553	0.370	0.471	0.386	0.525	0.403
10	0.538	0.455	0.392	0.199	0.136	0.026	0.065	0.023	0.067	0.023
15	0.344	0.173	0.210	0.038	0.026	0.006	0.004	0.000	0.003	0.000
20	0.205	0.046	0.115	0.009	0.005	0.001	0.000	0.000	0.000	0.000
25	0.118	0.010	0.067	0.002	0.001	0.000	0.000	0.000	0.000	0.000
30	0.073	0.002	0.038	0.000	0.000	0.000	0.000	0.000	0.000	0.000
					$\mu = 6$					
5	0.779	0.748	0.755	0.674	0.659	0.487	0.532	0.490	0.660	0.517
10	0.631	0.526	0.483	0.244	0.165	0.032	0.079	0.033	0.119	0.035
15	0.427	0.194	0.263	0.035	0.031	0.022	0.006	0.001	0.007	0.000
20	0.255	0.043	0.140	0.011	0.005	0.013	0.000	0.000	0.000	0.000
25	0.145	0.012	0.074	0.004	0.001	0.002	0.000	0.000	0.000	0.000
30	0.081	0.003	0.042	0.001	0.000	0.000	0.000	0.000	0.000	0.000

Table 2. Power adjusted size of tests, AR errors, $\Phi = -0.5$

c	S_0	$S_0(\mathbf{c})$	S_2	$S_2(\mathbf{c})$	S_1	$S_1(\mathbf{c})$	S_{12}	$S_{12}(\mathbf{c})$	S_{012}	$S_{012}(\mathbf{c})$
					$\mu = 0$					
5	0.560	0.558	0.398	0.351	0.190	0.207	0.093	0.127	0.083	0.117
10	0.350	0.287	0.218	0.101	0.036	0.016	0.006	0.006	0.004	0.004
15	0.193	0.086	0.108	0.014	0.005	0.000	0.000	0.000	0.000	0.000
20	0.098	0.015	0.050	0.001	0.001	0.000	0.000	0.000	0.000	0.000
25	0.048	0.001	0.022	0.000	0.000	0.000	0.000	0.000	0.000	0.000
30	0.021	0.000	0.008	0.000	0.000	0.000	0.000	0.000	0.000	0.000
					$\mu = 2$					
5	0.667	0.644	0.602	0.512	0.475	0.297	0.394	0.288	0.409	0.285
10	0.467	0.344	0.325	0.124	0.092	0.017	0.033	0.009	0.028	0.007
15	0.267	0.096	0.155	0.014	0.011	0.002	0.001	0.000	0.001	0.000
20	0.134	0.015	0.069	0.001	0.001	0.000	0.000	0.000	0.000	0.000
25	0.062	0.002	0.029	0.000	0.000	0.000	0.000	0.000	0.000	0.000
30	0.028	0.000	0.010	0.000	0.000	0.000	0.000	0.000	0.000	0.000
					$\mu = 4$					
5	0.776	0.744	0.751	0.672	0.680	0.498	0.552	0.551	0.667	0.574
10	0.622	0.468	0.473	0.183	0.149	0.022	0.058	0.021	0.089	0.019
15	0.401	0.122	0.232	0.014	0.018	0.016	0.002	0.000	0.003	0.000
20	0.213	0.015	0.101	0.003	0.001	0.013	0.000	0.000	0.000	0.000
25	0.098	0.002	0.041	0.001	0.000	0.006	0.000	0.000	0.000	0.000
30	0.043	0.001	0.015	0.000	0.000	0.002	0.000	0.000	0.000	0.000
					$\mu = 6$					
5	0.840	0.810	0.826	0.765	0.785	0.673	0.599	0.742	0.793	0.770
10	0.719	0.575	0.585	0.266	0.176	0.037	0.070	0.046	0.188	0.047
15	0.516	0.164	0.311	0.014	0.021	0.073	0.003	0.001	0.009	0.000
20	0.297	0.016	0.133	0.007	0.001	0.083	0.000	0.000	0.000	0.000
25	0.142	0.006	0.050	0.006	0.000	0.048	0.000	0.000	0.000	0.000
30	0.058	0.004	0.018	0.003	0.000	0.013	0.000	0.000	0.000	0.000

Table 3. Power adjusted size of tests, AR errors, $\Phi = -0.3$

c	S_0	$S_0(\mathbf{c})$	S_2	$S_2(\mathbf{c})$	S_1	$S_1(\mathbf{c})$	S_{12}	$S_{12}(\mathbf{c})$	S_{012}	$S_{012}(\mathbf{c})$
					$\mu = 0$					
5	0.560	0.571	0.399	0.359	0.187	0.204	0.090	0.123	0.080	0.116
10	0.354	0.305	0.224	0.115	0.039	0.016	0.007	0.006	0.005	0.004
15	0.204	0.102	0.118	0.019	0.006	0.000	0.000	0.000	0.000	0.000
20	0.111	0.021	0.062	0.002	0.001	0.000	0.000	0.000	0.000	0.000
25	0.059	0.002	0.032	0.000	0.000	0.000	0.000	0.000	0.000	0.000
30	0.030	0.000	0.015	0.000	0.000	0.000	0.000	0.000	0.000	0.000
					$\mu = 2$					
5	0.647	0.639	0.572	0.493	0.438	0.274	0.359	0.248	0.365	0.244
10	0.448	0.349	0.311	0.135	0.090	0.017	0.032	0.009	0.027	0.006
15	0.263	0.111	0.157	0.019	0.013	0.001	0.001	0.000	0.001	0.000
20	0.141	0.021	0.078	0.002	0.001	0.000	0.000	0.000	0.000	0.000
25	0.072	0.003	0.038	0.000	0.000	0.000	0.000	0.000	0.000	0.000
30	0.037	0.000	0.017	0.000	0.000	0.000	0.000	0.000	0.000	0.000
					$\mu = 4$					
5	0.751	0.725	0.724	0.642	0.635	0.442	0.523	0.470	0.614	0.485
10	0.589	0.456	0.441	0.181	0.144	0.021	0.061	0.017	0.078	0.016
15	0.375	0.131	0.222	0.018	0.021	0.009	0.003	0.000	0.003	0.000
20	0.206	0.020	0.105	0.004	0.002	0.005	0.000	0.000	0.000	0.000
25	0.103	0.003	0.049	0.001	0.000	0.001	0.000	0.000	0.000	0.000
30	0.052	0.001	0.022	0.000	0.000	0.000	0.000	0.000	0.000	0.000
					$\mu = 6$					
5	0.819	0.782	0.804	0.732	0.741	0.604	0.575	0.634	0.749	0.658
10	0.687	0.553	0.549	0.249	0.172	0.031	0.073	0.033	0.157	0.033
15	0.480	0.164	0.291	0.017	0.025	0.043	0.004	0.000	0.007	0.000
20	0.276	0.020	0.135	0.006	0.003	0.043	0.000	0.000	0.000	0.000
25	0.140	0.006	0.059	0.004	0.000	0.016	0.000	0.000	0.000	0.000
30	0.064	0.002	0.026	0.001	0.000	0.002	0.000	0.000	0.000	0.000

Table 4. Power adjusted size of tests, AR errors, $\Phi = 0.3$

c	S_0	$S_0(\mathbf{c})$	S_2	$S_2(\mathbf{c})$	S_1	$S_1(\mathbf{c})$	S_{12}	$S_{12}(\mathbf{c})$	S_{012}	$S_{012}(\mathbf{c})$
					$\mu = 0$					
5	0.567	0.640	0.403	0.424	0.189	0.277	0.098	0.250	0.090	0.258
10	0.387	0.444	0.255	0.226	0.055	0.053	0.013	0.044	0.008	0.045
15	0.264	0.260	0.166	0.093	0.016	0.004	0.002	0.005	0.001	0.004
20	0.182	0.126	0.116	0.030	0.005	0.000	0.000	0.000	0.000	0.000
25	0.132	0.050	0.085	0.006	0.002	0.000	0.000	0.000	0.000	0.000
30	0.098	0.014	0.062	0.001	0.001	0.000	0.000	0.000	0.000	0.000
					$\mu = 2$					
5	0.600	0.657	0.475	0.469	0.314	0.295	0.251	0.300	0.240	0.311
10	0.422	0.457	0.288	0.237	0.086	0.054	0.035	0.048	0.027	0.048
15	0.284	0.265	0.180	0.095	0.023	0.006	0.004	0.005	0.002	0.004
20	0.193	0.126	0.123	0.031	0.006	0.000	0.000	0.000	0.000	0.000
25	0.134	0.047	0.085	0.007	0.002	0.000	0.000	0.000	0.000	0.000
30	0.102	0.013	0.063	0.001	0.001	0.000	0.000	0.000	0.000	0.000
					$\mu = 4$					
5	0.659	0.693	0.591	0.548	0.462	0.347	0.406	0.386	0.422	0.403
10	0.491	0.494	0.348	0.250	0.128	0.054	0.069	0.056	0.061	0.057
15	0.329	0.268	0.209	0.094	0.034	0.011	0.007	0.005	0.005	0.004
20	0.219	0.122	0.135	0.032	0.009	0.002	0.001	0.000	0.000	0.000
25	0.148	0.045	0.094	0.010	0.003	0.000	0.000	0.000	0.000	0.000
30	0.106	0.013	0.066	0.002	0.001	0.000	0.000	0.000	0.000	0.000
					$\mu = 6$					
5	0.724	0.721	0.676	0.613	0.556	0.407	0.479	0.454	0.549	0.479
10	0.562	0.530	0.412	0.274	0.157	0.057	0.086	0.065	0.096	0.069
15	0.381	0.279	0.242	0.088	0.039	0.024	0.010	0.005	0.008	0.005
20	0.248	0.114	0.151	0.034	0.011	0.008	0.001	0.000	0.001	0.000
25	0.162	0.043	0.097	0.011	0.003	0.001	0.000	0.000	0.000	0.000
30	0.109	0.013	0.070	0.002	0.001	0.000	0.000	0.000	0.000	0.000

Table 5. Power adjusted size of tests, AR errors, $\Phi = 0.5$

c	S_0	$S_0(\mathbf{c})$	S_2	$S_2(\mathbf{c})$	S_1	$S_1(\mathbf{c})$	S_{12}	$S_{12}(\mathbf{c})$	S_{012}	$S_{012}(\mathbf{c})$
					$\mu = 0$					
5	0.574	0.675	0.407	0.467	0.190	0.342	0.111	0.354	0.099	0.370
10	0.408	0.526	0.275	0.303	0.062	0.106	0.017	0.112	0.010	0.115
15	0.301	0.372	0.194	0.174	0.022	0.019	0.003	0.022	0.001	0.022
20	0.226	0.235	0.149	0.088	0.009	0.002	0.001	0.004	0.000	0.004
25	0.177	0.133	0.120	0.036	0.004	0.000	0.000	0.000	0.000	0.000
30	0.144	0.060	0.097	0.010	0.002	0.000	0.000	0.000	0.000	0.000
					$\mu = 2$					
5	0.590	0.679	0.446	0.489	0.275	0.349	0.219	0.382	0.206	0.397
10	0.431	0.535	0.293	0.312	0.086	0.108	0.038	0.115	0.028	0.120
15	0.309	0.376	0.202	0.175	0.030	0.023	0.007	0.025	0.003	0.025
20	0.231	0.235	0.154	0.089	0.010	0.003	0.001	0.004	0.000	0.004
25	0.176	0.128	0.119	0.035	0.005	0.000	0.000	0.001	0.000	0.000
30	0.145	0.056	0.096	0.011	0.003	0.000	0.000	0.000	0.000	0.000
					$\mu = 4$					
5	0.627	0.698	0.529	0.538	0.396	0.370	0.356	0.430	0.351	0.447
10	0.470	0.553	0.329	0.315	0.123	0.107	0.072	0.119	0.058	0.124
15	0.335	0.373	0.218	0.174	0.041	0.030	0.012	0.026	0.007	0.025
20	0.242	0.231	0.156	0.086	0.015	0.007	0.002	0.004	0.000	0.004
25	0.183	0.119	0.123	0.037	0.006	0.001	0.000	0.000	0.000	0.000
30	0.144	0.049	0.098	0.010	0.003	0.000	0.000	0.000	0.000	0.000
					$\mu = 6$					
5	0.681	0.715	0.605	0.584	0.476	0.400	0.434	0.474	0.460	0.494
10	0.518	0.570	0.371	0.329	0.151	0.104	0.093	0.125	0.086	0.133
15	0.366	0.378	0.238	0.168	0.047	0.043	0.015	0.027	0.010	0.026
20	0.259	0.219	0.168	0.086	0.017	0.013	0.003	0.005	0.001	0.004
25	0.189	0.107	0.124	0.035	0.006	0.001	0.001	0.000	0.000	0.000
30	0.142	0.040	0.098	0.008	0.002	0.000	0.000	0.000	0.000	0.000

Table 6. Power adjusted size of tests, MA errors, $\Theta = -0.5$

c	S_0	$S_0(\mathbf{c})$	S_2	$S_2(\mathbf{c})$	S_1	$S_1(\mathbf{c})$	S_{12}	$S_{12}(\mathbf{c})$	S_{012}	$S_{012}(\mathbf{c})$
					$\mu = 0$					
5	0.564	0.564	0.403	0.381	0.215	0.251	0.115	0.157	0.106	0.146
10	0.352	0.301	0.201	0.115	0.038	0.025	0.007	0.008	0.006	0.006
15	0.186	0.096	0.087	0.019	0.004	0.001	0.000	0.000	0.000	0.000
20	0.089	0.017	0.036	0.002	0.001	0.000	0.000	0.000	0.000	0.000
25	0.040	0.002	0.015	0.000	0.000	0.000	0.000	0.000	0.000	0.000
30	0.018	0.000	0.005	0.000	0.000	0.000	0.000	0.000	0.000	0.000
					$\mu = 2$					
5	0.702	0.674	0.664	0.580	0.584	0.385	0.505	0.397	0.531	0.402
10	0.508	0.378	0.358	0.152	0.116	0.028	0.041	0.014	0.041	0.011
15	0.292	0.108	0.154	0.018	0.011	0.006	0.001	0.000	0.001	0.000
20	0.140	0.018	0.059	0.002	0.001	0.002	0.000	0.000	0.000	0.000
25	0.061	0.002	0.022	0.001	0.000	0.001	0.000	0.000	0.000	0.000
30	0.029	0.001	0.008	0.000	0.000	0.000	0.000	0.000	0.000	0.000
					$\mu = 4$					
5	0.813	0.778	0.801	0.734	0.806	0.618	0.713	0.692	0.803	0.726
10	0.679	0.521	0.550	0.240	0.214	0.038	0.076	0.041	0.151	0.039
15	0.459	0.147	0.277	0.019	0.021	0.054	0.002	0.001	0.005	0.000
20	0.255	0.020	0.110	0.007	0.001	0.060	0.000	0.000	0.000	0.000
25	0.120	0.005	0.040	0.006	0.000	0.043	0.000	0.000	0.000	0.000
30	0.056	0.003	0.013	0.003	0.000	0.021	0.000	0.000	0.000	0.000
					$\mu = 6$					
5	0.870	0.838	0.865	0.813	0.897	0.777	0.797	0.854	0.904	0.882
10	0.769	0.630	0.665	0.351	0.278	0.066	0.092	0.097	0.306	0.106
15	0.588	0.208	0.397	0.018	0.028	0.188	0.003	0.005	0.022	0.004
20	0.377	0.021	0.172	0.019	0.002	0.230	0.000	0.004	0.001	0.001
25	0.199	0.013	0.058	0.027	0.000	0.190	0.000	0.000	0.000	0.000
30	0.093	0.013	0.019	0.021	0.000	0.115	0.000	0.000	0.000	0.000

Table 7. Power adjusted size of tests, MA errors, $\Theta = -0.3$

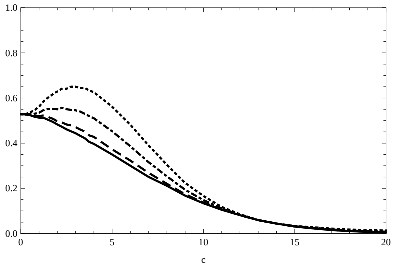
c	S_0	$S_0(\mathbf{c})$	S_2	$S_2(\mathbf{c})$	S_1	$S_1(\mathbf{c})$	S_{12}	$S_{12}(\mathbf{c})$	S_{012}	$S_{012}(\mathbf{c})$
					$\mu = 0$					
5	0.559	0.566	0.397	0.362	0.189	0.214	0.092	0.129	0.082	0.121
10	0.350	0.302	0.215	0.114	0.037	0.018	0.007	0.006	0.005	0.004
15	0.195	0.100	0.107	0.019	0.005	0.000	0.000	0.000	0.000	0.000
20	0.102	0.021	0.052	0.002	0.001	0.000	0.000	0.000	0.000	0.000
25	0.052	0.002	0.026	0.000	0.000	0.000	0.000	0.000	0.000	0.000
30	0.027	0.000	0.011	0.000	0.000	0.000	0.000	0.000	0.000	0.000
					$\mu = 2$					
5	0.658	0.644	0.591	0.509	0.464	0.298	0.381	0.273	0.393	0.272
10	0.456	0.352	0.316	0.137	0.091	0.019	0.033	0.009	0.028	0.007
15	0.264	0.109	0.151	0.019	0.012	0.001	0.001	0.000	0.001	0.000
20	0.137	0.021	0.070	0.002	0.001	0.000	0.000	0.000	0.000	0.000
25	0.067	0.003	0.032	0.000	0.000	0.000	0.000	0.000	0.000	0.000
30	0.034	0.001	0.014	0.000	0.000	0.000	0.000	0.000	0.000	0.000
					$\mu = 4$					
5	0.765	0.736	0.741	0.661	0.675	0.480	0.557	0.518	0.654	0.536
10	0.607	0.468	0.462	0.191	0.153	0.023	0.061	0.019	0.086	0.019
15	0.390	0.132	0.228	0.019	0.020	0.014	0.002	0.000	0.003	0.000
20	0.211	0.020	0.101	0.004	0.002	0.010	0.000	0.000	0.000	0.000
25	0.102	0.004	0.045	0.002	0.000	0.004	0.000	0.000	0.000	0.000
30	0.051	0.001	0.019	0.000	0.000	0.000	0.000	0.000	0.000	0.000
					$\mu = 6$					
5	0.831	0.792	0.819	0.750	0.786	0.645	0.616	0.692	0.786	0.719
10	0.705	0.568	0.574	0.267	0.186	0.036	0.075	0.039	0.178	0.042
15	0.502	0.171	0.308	0.017	0.024	0.064	0.004	0.000	0.009	0.000
20	0.293	0.020	0.136	0.008	0.002	0.072	0.000	0.000	0.000	0.000
25	0.148	0.007	0.056	0.006	0.000	0.036	0.000	0.000	0.000	0.000
30	0.068	0.003	0.023	0.002	0.000	0.007	0.000	0.000	0.000	0.000

Table 8. Power adjusted size of tests, MA errors, $\Theta = 0.3$

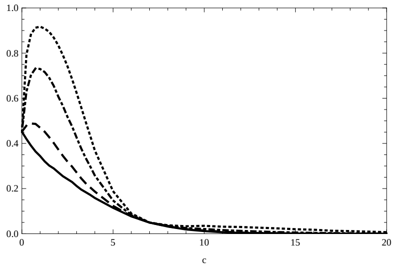
c	S_0	$S_0(\mathbf{c})$	S_2	$S_2(\mathbf{c})$	S_1	$S_1(\mathbf{c})$	S_{12}	$S_{12}(\mathbf{c})$	S_{012}	$S_{012}(\mathbf{c})$
					$\mu = 0$					
5	0.564	0.627	0.401	0.409	0.188	0.263	0.093	0.220	0.087	0.227
10	0.379	0.421	0.246	0.203	0.051	0.045	0.011	0.032	0.007	0.032
15	0.251	0.234	0.154	0.076	0.014	0.003	0.001	0.003	0.001	0.002
20	0.165	0.108	0.102	0.022	0.004	0.000	0.000	0.000	0.000	0.000
25	0.116	0.040	0.071	0.004	0.001	0.000	0.000	0.000	0.000	0.000
30	0.083	0.011	0.051	0.000	0.000	0.000	0.000	0.000	0.000	0.000
					$\mu = 2$					
5	0.603	0.651	0.484	0.462	0.326	0.286	0.259	0.277	0.250	0.286
10	0.419	0.437	0.284	0.215	0.083	0.046	0.032	0.036	0.026	0.036
15	0.274	0.240	0.171	0.077	0.021	0.005	0.003	0.003	0.002	0.002
20	0.179	0.107	0.110	0.022	0.004	0.000	0.000	0.000	0.000	0.000
25	0.120	0.038	0.072	0.005	0.002	0.000	0.000	0.000	0.000	0.000
30	0.089	0.010	0.052	0.001	0.001	0.000	0.000	0.000	0.000	0.000
					$\mu = 4$					
5	0.669	0.692	0.611	0.555	0.484	0.348	0.420	0.378	0.444	0.395
10	0.498	0.479	0.355	0.232	0.128	0.046	0.065	0.043	0.058	0.044
15	0.327	0.245	0.203	0.075	0.031	0.010	0.006	0.003	0.004	0.002
20	0.209	0.104	0.124	0.024	0.008	0.002	0.000	0.000	0.000	0.000
25	0.135	0.036	0.083	0.007	0.002	0.000	0.000	0.000	0.000	0.000
30	0.094	0.011	0.055	0.001	0.001	0.000	0.000	0.000	0.000	0.000
					$\mu = 6$					
5	0.737	0.725	0.695	0.623	0.587	0.418	0.494	0.453	0.576	0.479
10	0.576	0.522	0.427	0.260	0.159	0.050	0.082	0.053	0.098	0.057
15	0.387	0.256	0.242	0.070	0.037	0.023	0.008	0.003	0.008	0.003
20	0.244	0.097	0.144	0.027	0.008	0.009	0.001	0.000	0.000	0.000
25	0.154	0.036	0.088	0.009	0.002	0.001	0.000	0.000	0.000	0.000
30	0.100	0.011	0.060	0.002	0.001	0.000	0.000	0.000	0.000	0.000

Table 9. Power adjusted size of tests, MA errors, $\Theta = 0.5$

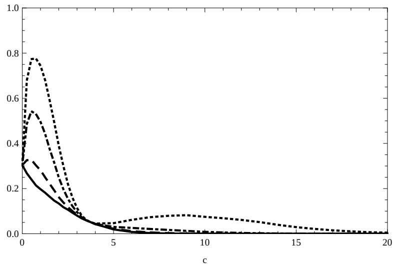
c	S_0	$S_0(\mathbf{c})$	S_2	$S_2(\mathbf{c})$	S_1	$S_1(\mathbf{c})$	S_{12}	$S_{12}(\mathbf{c})$	S_{012}	$S_{012}(\mathbf{c})$
					$\mu = 0$					
5	0.566	0.636	0.402	0.422	0.190	0.286	0.095	0.254	0.089	0.263
10	0.384	0.445	0.250	0.228	0.053	0.061	0.012	0.048	0.008	0.049
15	0.259	0.272	0.160	0.100	0.015	0.007	0.002	0.005	0.001	0.005
20	0.176	0.145	0.109	0.037	0.005	0.000	0.000	0.001	0.000	0.000
25	0.127	0.069	0.079	0.011	0.002	0.000	0.000	0.000	0.000	0.000
30	0.095	0.027	0.059	0.002	0.001	0.000	0.000	0.000	0.000	0.000
					$\mu = 2$					
5	0.595	0.650	0.468	0.463	0.303	0.303	0.235	0.300	0.225	0.310
10	0.415	0.456	0.281	0.238	0.080	0.064	0.031	0.051	0.024	0.051
15	0.276	0.275	0.173	0.101	0.021	0.008	0.003	0.006	0.002	0.005
20	0.185	0.145	0.115	0.037	0.005	0.001	0.000	0.000	0.000	0.000
25	0.129	0.066	0.080	0.012	0.002	0.000	0.000	0.000	0.000	0.000
30	0.100	0.026	0.060	0.003	0.001	0.000	0.000	0.000	0.000	0.000
					$\mu = 4$					
5	0.652	0.686	0.580	0.539	0.452	0.349	0.392	0.380	0.402	0.397
10	0.481	0.487	0.339	0.249	0.122	0.062	0.063	0.058	0.054	0.060
15	0.318	0.279	0.200	0.100	0.031	0.015	0.007	0.006	0.004	0.005
20	0.210	0.141	0.127	0.038	0.008	0.003	0.001	0.001	0.000	0.000
25	0.142	0.062	0.089	0.014	0.003	0.000	0.000	0.000	0.000	0.000
30	0.103	0.024	0.063	0.004	0.001	0.000	0.000	0.000	0.000	0.000
					$\mu = 6$					
5	0.715	0.715	0.666	0.602	0.551	0.402	0.473	0.449	0.532	0.472
10	0.549	0.521	0.403	0.271	0.153	0.065	0.081	0.066	0.088	0.072
15	0.369	0.287	0.233	0.094	0.037	0.028	0.009	0.006	0.008	0.006
20	0.240	0.134	0.145	0.041	0.010	0.010	0.001	0.001	0.000	0.000
25	0.158	0.058	0.093	0.016	0.003	0.001	0.000	0.000	0.000	0.000
30	0.108	0.022	0.068	0.004	0.001	0.000	0.000	0.000	0.000	0.000



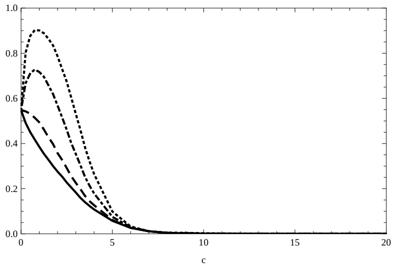
(a) S_0: $\bar{\alpha}_1 = |\mu|$



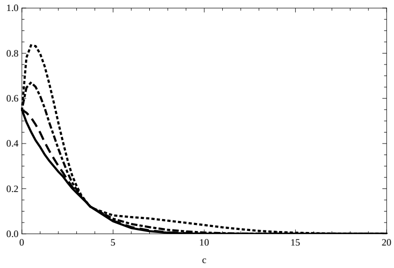
(b) S_2: $\bar{\alpha}_2 = |\mu|$



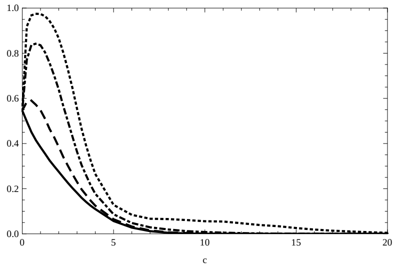
(c) S_1: $\bar{\alpha}_3 = \bar{\alpha}_4 = |\mu|$



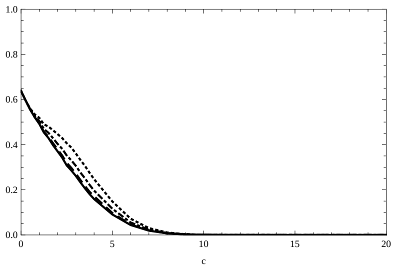
(d) S_{12}: $\bar{\alpha}_2 = |\mu|$, $\bar{\alpha}_3 = \bar{\alpha}_4 = 0$



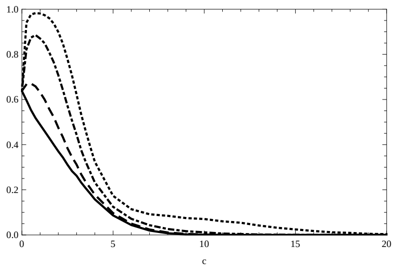
(e) S_{12}: $\bar{\alpha}_2 = 0$, $\bar{\alpha}_3 = \bar{\alpha}_4 = |\mu|$



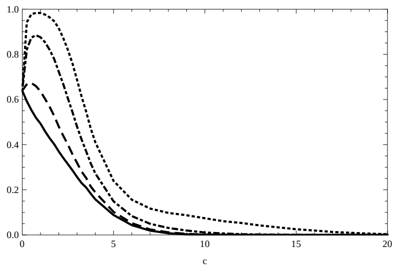
(f) S_{12}: $\bar{\alpha}_2 = \bar{\alpha}_3 = \bar{\alpha}_4 = |\mu|$



(g) S_{012}: $\bar{\alpha}_1 = |\mu|$,
$\bar{\alpha}_2 = \bar{\alpha}_3 = \bar{\alpha}_4 = 0$



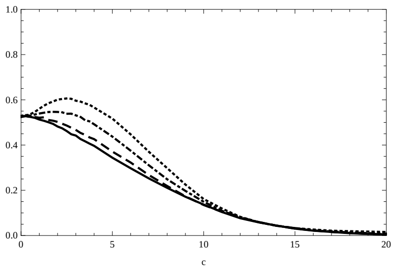
(h) S_{012}: $\bar{\alpha}_1 = \bar{\alpha}_2 = 0$,
$\bar{\alpha}_3 = \bar{\alpha}_4 = |\mu|$



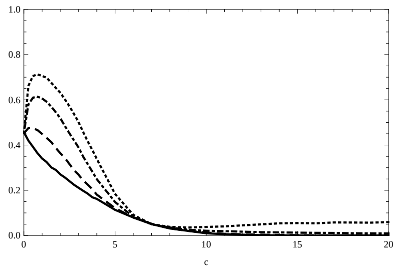
(i) S_{012}:
$\bar{\alpha}_1 = \bar{\alpha}_2 = \bar{\alpha}_3 = \bar{\alpha}_4 = |\mu|$

Figure 1. Asymptotic size, fixed initial condition

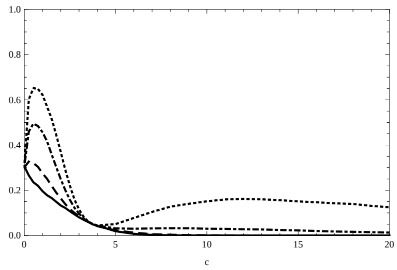
$\mu = 0$: ——— , $\mu = 2$: — — , $\mu = 4$: — · —, $\mu = 6$: · · ·



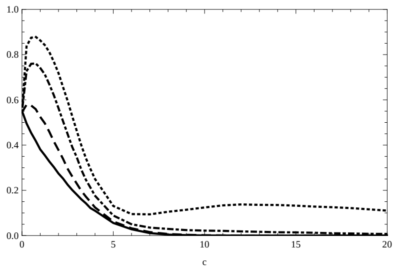
(a) S_0



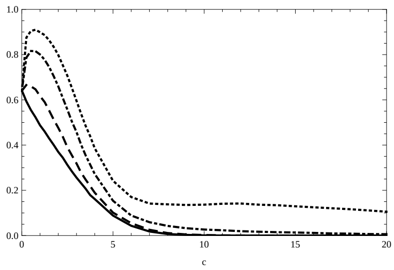
(b) S_2



(c) S_1



(d) S_{12}



(e) S_{012}

Figure 2. Asymptotic size, random initial condition, $\alpha_i \sim N(0, \sigma_\alpha^2)$, $i = 1, 2, 3, 4$

$\mu = 0$: ——— , $\mu = 2$: — — , $\mu = 4$: — · —, $\mu = 6$: · · ·