# MODELLING SPOT RATE PROCESS IN THE RUSSIAN TREASURY BILLS MARKET

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## Abstract

The paper deals with modelling of spot rate process in the market for government securities in transitional economy. The case of the Russian Treasury bills market is taken as an example. We use three approaches to estimation of parameters of spot rate stochastic process: AR-GARCH time series models, GMM estimates and stochastic volatility models (QML estimates and Kalman filter). The most general conclusion is that pattern of spot rate process in transitional economy can be nested in existing theoretical model of term structure of interest rates. Estimated parameters of the spot rate process indicate that the Russian market for government securities by its features is closer to the European financial markets compared to the market for US Treasury bills. This conclusion is supported by estimates of parameters of the GKO spot rate stochastic process using both the GMM and QML estimates of spot rate nonlinear models. The Cox-Ingersoll-Ross 1985 model of term structure of interest rates is the most adequate for the Russian GKO market. The behaviour of the term structure of GKO yields in 1994 through 1998 did not contradict to theoretical conclusions from the model; analytical yield curves have satisfactory accuracy of approximation of actual GKO yield curves. The spot rate stochastic process is mean-reverting, but its variance although being stochastic does not exhibit mean-reverting property (according to Kalman filter estimates). The stochastic nature of spot rate volatility origins from different responses to ‘good’ and ‘bad’ news and a proportion to current spot rate level (but less than one by one).

## Introduction

The goal of the study is modelling of spot rate process in the market for government securities in transitional economy. The case of the Russian Treasury bills market is taken as an example. We investigated the period before the Russian financial crisis in August 1998, as in its aftermath the market lost liquidity and has not recovered yet. A continuous time series on yields are available only for the pre-crisis period.

Specification of short-term spot rate stochastic process is a starting point in analysing term structure of interest rates. In a number of term structure models, spot rate is taken as state variable determined by economic fundamentals. Information on the form of spot rate process is sufficient to describe behaviour of the whole term structure of interest rats. Analytical yield curves in any time can be uniquely derived from spot rate value at the given time. Since an exact form of the function is unknown in advanced, evaluation of stochastic process parameters has to be done empirically on basis of comparison across alternative theoretical and econometric specifications.

A great number of theoretical models o f term structure of interest rates have been developed in economic literature. We conditionally divided the models into four types[1]:

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[1] In addition, there are a number of term structure models based on interpolation of actual yield curves using different types of smooth functions (e.g., Vasicek, O., H. G. Fong (1982) ‘Term structure modeling using exponential splines’, *Journal of Finance*, 37, pp. 339 – 348; Nelson, C., A. Siegel (1987) 'Parsimonious modeling of yield curve', *Journal of Business*, 60, pp. 473 – 489; Anderson, N., F. Breedon, M. Deacon, A. Derry, G. Murphy (1996) *Estimating and Interpreting the Yield Curve*. John Wiley & Sons Ltd.; Ferguson, R., S. Raymar (1998) 'A comparative analysis of several popular term structure estimation models', *Journal of Fixed Income*, 7, pp. 17 - 33).

macroeconomic approaches[2], factor stochastic models[3], general equilibrium stochastic models[4], arbitrage-free models[5].

In the present study, the modelling of stochastic process of spot interest rates on the shortest Russian Treasury bills (GKO in Russian abbreviation) is based on different versions of factor and general equilibrium stochastic models. These models are aimed at deriving an term structure of interest rates on the basis of market-clearing assumptions and finding out a form of spot rate stochastic process, which can generate a plausible equilibrium term structure of interest rates. We do not consider macroeconomic approaches and arbitrage-free models, since those models are aimed at analysing of impact of economic policy shocks and solving derivative pricing problem, correspondingly, but are less suitable for solving our task.

## §1. Data

The initial data for the study were taken from the database of the Russian Informational Agency *Fimarket*. We chose the yield to maturity on GKOs of one-week maturity (in annual terms) as an indicator of short-term spot rate at the Russian market for government securities (further – GKO spot rate, $r_t$). We consider weekly data for the period from the 12[th] of September 1994 through the 14[th] August 1998. The total number of observations is 205. The spot rate is calculated as continuously-compounded GKO yield to maturity, i.e.

$$r_t = \frac{-\ln P_t}{T/365},$$

where $P_t$ – bill price in portions of unity.

In January 1997 the Russian State Duma adopted a law on taxation of income on government securities. That innovation should apparently affect prices of new issued bills. The discount was taxed at fixed rate of 15%. Calculating yields on taxed GKO issues, we suppose that the bill is held all the term. Hence, the tax deducted is 15% of discount or

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[2] E.g., O.Blanchard, 'Output, the stock market, and interest rates', *American Economic Review*, 71, 1981, pp. 132–143; S.Turnovsky, 'The term structure of interest rates and the effects of macroeconomic policy', *Journal of Money, Credit and Banking*, 21, 1989, pp. 321–347.

[3] E.g., O.Vasicek, 'An equilibrium characterization of the term structure', *Journal of Financial Economics*, 5, 1977, pp. 177–188; U.Dothan, 'On the term structure of interest rates', *Journal of Financial Economics*, 6, 1978, pp. 59–69.

[4] E.g., J.Cox, J.Ingersoll, S.Ross, 'A theory of the term structure of interest rates', *Econometrica*, 53, 1985, pp. 385–407; F.Longstaff, E.Schwartz, 'Interest rate volatility and the term structure: A two-factor general equilibrium model', *Journal of Finance*, 47, 1992, pp. 1259–1282.

[5] E.g., T.Ho, Sang-Bin Lee, 'Term structure movements and pricing interest rate contingent claims', *Journal of Finance*, 41, 1986, pp. 1011–1029; D.Heath, R.Jarrow, A.Morton, 'Bond pricing and the term structure of interest rates: A new methodology for contingent claims valuation', *Econometrica*, 60, 1992, pp. 77–105.

increment of bill price between the time of issuing and current date. For uniformity of continuous time series of yields the tax is added to the market price and the sum is used to calculate compounded yield, i.e.

$$r_t = \frac{-\ln \tilde{P}_t}{T/365} .$$

$$\tilde{P}_t = P_t + 0{,}15 \cdot (100 - P_t)$$

The average-weighted spot rate for a week is determined by the formula:

$$r_t^a = \frac{1}{V} \sum_t r_t V_t \ ,$$

where $V$ – total trading volume on bills of maturity less than a week at secondary market during the current week, $r_t$ – continuously-compounded yield at day $t$, $V_t$ – trading volume on bill of maturity less than a week at day $t$.

Modelling of the GKO spot rate on the Russian government short-term bills meets certain difficulties and takes into account some assumptions which are to be specifically noted.

***First,*** the time period of observation is less than four years. Thus, we review the data with week frequency in order to ensure a sufficient number of degrees of freedom in econometric models. At the same time, the majority of studies of term structure of interest rates and spot rate processes on the Western markets are based on monthly or quarterly data. The higher frequency of observations may cause stronger influence of random noises and fluctuations in the market related rather to short-term variations of market liquidity, actions of individual large participants than to macroeconomic factors.

***Second,*** high-frequent observations lead to gaps in actually observed data. Except of period between the second half of 1996 to 1998, one-week bills did not exist every week. In order to create continuous time series of the GKO spot rate we approximated the mixed values using adjacent observations.

***Third***, a majority of theoretical models of term structure is either based on the analysis of real interest rates, but, empirical studies of term structure of yields at developed and emerging markets deal with nominal rates. This is chiefly related to the fact that in developed countries inflation is low over short (monthly, quarterly) periods, and the transition to real *ex post* rates affects the general market pattern insignificantly, while the economic interpretation of real *ex post* rates is ambiguous. In case the hypothesis that the real rate is constant (in short run) is applied, the variation of inflationary expectations and risk premium randomise the dynamics of nominal interest rates thus making possible to model them as a stochastic process. Therefore, this study also deals with modelling of stochastic nominal GKO spot rate.

## §2. Methodology of spot rate stochastic process estimation

In theoretical models of term structure of interest rates a short-term spot rate is an indicator of instantaneous discount rate (marginal intertemporal substitution rate) for the market participants. Evidently, its dynamics should be rather smooth, its fluctuations are caused by changes in economic agents' preferences which could not alter too often. The assumed smoothing of spot rate process adds some inaccuracy in estimation, and goodness-of-fit statistics are usually low in the models.

Analysing term structure of yields on government securities, one should turn to the problem of division between nominal and real variables. All theoretical stochastic models of term structure of interest rates consider real riskless rate of interest. Since the term structure of real (*ex ante*) interest rates is not observable at each moment of time, empirical studies of yield curve take two different approaches.

According to the first one proposed by Brown and Dybvig[6], a stochastic model of term structure of real interest rates is directly transformed in a model of nominal interest rates. The authors assume that all properties of theoretical yield curve and inferences from the initial model are valid for nominal interest rates[7]. In addition, nominal bond yields definitely cannot be negative; the latter corresponds to a number of theoretical models (e.g., the Cox-Ingersoll-Ross model).

The second approach is based on building a multifactor model. Besides the equation for nominal spot rate process, one consider a process of inflation rate or another indicator (e.g., exchange rate, which allows to analyse term structure of interest rates in real terms. This approach was used, e.g., by Richard and Dillen[8].

To simplify the estimation, we follow the first approach, which is the most prevailing in empirical studies of financial markets in developed countries.

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[6] Brown, S., P. Dybvig (1986) 'The empirical implications of the Cox, Ingersoll, Ross theory of the term structure of interest rates', *Journal of Finance*, 41, pp. 616 – 630.

[7] I.e. components of nominal spot rate stochastic process corresponding to real *ex ante* rate and those of inflation process are independent (see Campbell, Lo, MacKinlay, 1997).

[8] Richard, S. (1978) 'An arbitrage model of the term structure of interest rates', *Journal of Financial Economics*, 6, pp. 33 – 57; Dillen, H. (1997) 'A model of the term structure of interest rates in an open economy with regime shifts', *Journal of International Money and Finance*, 16, pp. 795 – 819.

There are a number of alternative methods of estimation of parameters of a spot rate stochastic process:

1. Nonlinear parametric models estimated using maximum likelihood method[9];
2. Nonlinear time series models with autoregressive conditional heteroskedasticity in residuals;
3. Nonlinear parametric models estimated using generalised method of moments;
4. Gaussian models[10];
5. Nonparametric models[11];
6. Stochastic volatility models[12].

In the study we use the most widespread second, third and sixth approaches to estimation of parameters of spot rate stochastic process. Nonlinear time series models of short-term spot rates with autoregressive conditional heteroskedasticity in residuals are not straightforward related to any specification of spot rate stochastic process derived in theoretical term structure models. However, well-developed and widely known mathematical tools allow to consider this approach as one of the main methods of estimation of parameters of spot rate stochastic process. Fornari and Mele[13] compared econometric (on the basis of ARCH-GARCH models) and theoretical specification of the process. They showed that, despite formal difference in notation of theoretical and econometric equation, comparative estimation of different specifications of conditional variances can reveal the main properties

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[9] See, e.g., Brown, S., P. Dybvig (1986) 'The empirical implications of the Cox, Ingersoll, Ross theory of the term structure of interest rates', *Journal of Finance*, 41, pp. 616 – 630; Brown, R., S. Schaefer (1994) 'The term structure of real interest rates and the Cox, Ingersoll, and Ross model', *Journal of Financial Economics*, 35, pp. 3 – 42; Pearson, N., Tong-Sheng Sun (1994) 'Exploiting the conditional density in estimating the term structure: An application to the Cox, Ingersoll, and Ross model', *Journal of Finance*, 49, pp. 1279 – 1304; Overbeck, L., T. Ryden (1997) 'Estimation in the Cox-Ingersoll-Ross model', *Econometric Theory*, 13, pp. 430 – 461; De Munnik, J., P. Schotman (1994) 'Cross-section versus time series estimation of term structure models: Empirical results for the Dutch bond market', *Journal of Banking and Finance*, 18, pp. 997 – 1025.

[10] See, e.g., Bergstrom, A. (1990) *Continuous Time Econometric Modelling*. Oxford: Oxford University Press; Kennedy, D. P. (1997) 'Characterizing Gaussian models of the term structure of interest rates', *Mathematical Finance*, 7, pp. 107 – 118; Nowman, K. (1997) 'Gaussian estimation of single-factor continuous time models of the term structure of interest rates', *Journal of Finance*, 52, pp. 1695 – 1706.

[11] See, e.g., Jiang, G., J. Knight (1997) 'A nonparametric approach to the estimation of diffusion processes, with an application to a short-term interest rate model', *Econometric Theory*, 13, pp. 615 – 645; Jiang, G. (1998) 'Nonparametric modeling of U.S. interest rate term structure dynamics and implications on the prices of derivative securities', *Journal of Financial and Quantitative Analysis*, 33, pp. 465 - 497. Nonparametric estimation of term structure of GKO yields in 1995–1997 were made by (Novikov, D. Modeling the demand for Russian government securities from non-residents. – Moscow: NES, 1999).

[12] See, e.g., Ait-Sahalia, Y. (1996) 'Testing continuous-time models of the spot interest rate', Review of Financial Studies, 9, pp. 385–426; Ball, C., W. Torous (1999) 'The stochastic volatility of short-term interest rates: Some international evidence', *Journal of Finance*, 54, pp. 2339–2359.

[13] Fornari, F., A. Mele (1995) 'Continuous time conditionally heteroskedastic models: Theory with applications to the term structure of interest rates, *Economic Notes (Banca Monte dei Paschi di Siena SpA)*, 24, pp. 327 – 352.

of spot rate stochastic process and draw a conclusion on adequacy of assumed theoretical form of the process.

In 1992 Chan, Karolyi, Longstaff and Sanders[14] published a paper with empirical comparison of alternative models of short-term spot rate using general method of moments estimates. Later, this method was used by many authors carrying out comparative studies of alternative spot rate stochastic processes[15]. The Generalised Method of Moments (GMM) was proposed by Hansen in 1982[16]. This method has a number of advantages, which make it the most appropriate for estimation of continuous spot rate processes.

***First***, the GMM does not require normal distribution of changes in spot rate process. Asymptotically, the sufficient condition is stationarity and ergodicity of time series and existence of sufficient number of first moments. This property is of particular importance in our case, since each theoretical model assumes another distribution form of continuous spot rate process.

***Second***, GMM estimates and their standard errors are consistent even if residuals are heteroskedastic. Since estimating a continuous process using discrete observation, one meets a problem of data aggregation through time, which influenced the form of residuals distribution, that GMM property allows to reduce an impact of discrete approximation on standard errors of parameters estimates.

However, the GMM can be applied only to *large* samples, i.e. the aforementioned properties can be attained at large number of observations. In most cases GMM estimates are asymptotically efficient, but, they are hardly efficient at finite samples[17].

An alternative way to estimate a continuous spot rate process without loss of efficiency and consistency of estimates is to use so-called stochastic volatility models. The quasi maximum likelihood method is an appropriate technique for estimation. The seminal papers dealing with the approach are by Ait-Sahalia and Ball and Torous[18]. The authors

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[14] Chan, K., G. A. Karolyi, F. Longstaff, A. Sanders (1992) 'An empirical comparison of alternative models of the short-term interest rate', *Journal of Finance*, 47, pp. 1209 – 1227.

[15] E.g., Boero, G., C. Torricelli (1996) 'A comparative evaluation of alternative models of the term structure of interest rates', *European Journal of Operational Research*, 93, pp. 205 – 223; Vetzal, K. (1997) 'Stochastic volatility, movements in short term interest rates, and bond option value', *Journal of Banking and Finance*, 21, pp. 169 - 196; Raj, M., A. B. Sim, D. Thurston (1997) 'A generalized method of moments comparison of the Cox-Ingersoll-Ross and Heath-Jarrow-Morton models', *Journal of Economics and Business*, 49, pp. 169 – 192.

[16] Hansen, L. P. (1982) 'Large sample properties of Generalised Method of Moments estimators', *Econometrica*, 50, pp. 1029 – 1054.

[17] Discussion on GMM properties and its applications to financial markets see in Campbell, J., A. Lo, A. C. MacKinlay (1997) *The Econometrics of Financial Markets*. Princeton: Princeton University Press.

[18] Ait-Sahalia, Y. (1996) 'Testing continuous-time models of the spot interest rate', *Review of Financial Studies*, 9, pp. 385–426; Ball, C., W. Torous (1999) 'The stochastic volatility of short-term interest rates: Some international evidence', *Journal of Finance*, 54, pp. 2339–2359.

proposed several econometric specifications (including one similar to that used with GMM estimates and based on Kalman filter). An important advantage of the QML method is that it produces estimates efficient and consistent estimates even at small samples. Ait-Sahalia proposed a very general form of stochastic spot rate process which has no economic explanation or might be deduced from any theoretical model, thus we disregard the specification.

In the study, to estimate parameters of alternative GKO spot rate processes, we use methods which are similar to those developed by Fornari-Mele, Chan, Karolyi, Longstaff and Sanders and Ball and Torous.

## §3. Estimates of nonlinear time series models of spot rate with autoregressive conditional heteroskedasticity in residuals

Dickey-Fuller, Philips-Perron and Enders-Granger unit root tests (the latter is the test on joint hypothesis on presence of unit root and asymmetry in the process[19]) reject the hypotheses on presence of unit root and on asymmetry in the process for weekly time series of spot rate (see Table 1). The autocorrelation and partial autocorrelation function shown in Fig. 1 indicate that the spot rate is the second-order autoregressive process. Hence, we start from estimating a linear AR(2) model:

$$r_t = c + a_1 r_{t-1} + a_2 r_{t-2} + e_t \,. \qquad (1)$$

Estimates of equation (1) are given in Table 2. The Box-Ljung Statistics indicates that the residuals are not serially correlated, the Lagrange multiplier test does not reject the hypothesis on autoregressive conditional heteroskedasticity in residuals at 5% significance level (we omitted the results of both tests in the table). Therefore, we reveal the presence of conditional stochastic volatility in the GKO spot rate process. The next step is estimation of nonlinear models with conditional residual variance specified as different types of GARCH models.

We consider four specifications of equations of conditional residual variance including those presented in the paper by Fornari and Mele:

1) Generalised model GARCH(1,1)

$$s(e)_t^2 = d + a e_{t-1}^2 + b s_{t-1}^2 + h_t \qquad (2)$$

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[19] See Enders, W., C. Granger (1998) ‘Unit-root tests and asymmetric adjustment with an example using the term structure of interest rates’, *Journal of Business and Economic Statistics*, 16, pp. 304 – 311. The asymmetry of process means that interest rates are likely to deviate from long-term mean by a greater value to one side than to another one (e.g., more rise, than fall). That pattern could be responsible for non-stationarity of the process indicated by ordinary unit root tests.

2) Threshold model TARCH(1,1)

$$s(e)_t^2 = d + a e_{t-1}^2 + b s_{t-1}^2 + g e_{t-1}^2 d_{t-1} + h_t$$
$$d_t = \begin{cases} 1, e_t < 0 \\ 0, e_t \geq 0 \end{cases} \tag{3}$$

3) Exponential model EGARCH(1,1)

$$\log s(e)_t^2 = d + b \log s_{t-1}^2 + a \left| \frac{e_{t-1}}{s_{t-1}} \right| + g \frac{e_{t-1}}{s_{t-1}} + h_t \tag{4}$$

4) Component GARCH(1,1) model

$$\begin{aligned} s(e)_t^2 &= q_t + a(e_{t-1}^2 - q_{t-1}) + b(s_{t-1}^2 - q_{t-1}) + h_t \\ q_t &= d + g(q_{t-1} - d) + f(e_{t-1}^2 - s_{t-1}^2) \end{aligned} \tag{5}$$

Estimates of equation (1) with autoregressive conditional variance of residuals in the forms (2)–(5) are given in Table 2.

Our results enable to draw a number of conclusions concerning properties of the GKO spot rate stochastic process:

- Coefficients estimates of autoregressive terms in the main equation are positive in all models, the sums of coefficients are less than unity. Hence, the process is mean-reverting[20].
- Estimates of coefficients in variance equations $a + b > 1$ in models (2) and (3). Therefore, volatility of spot rate is non-stationary and rises with time[21]. The similar conclusion results from estimates of the component model, since the degree of convergence of temporary fluctuations of the variance to zero is $a + b \approx 1$.
- Asymmetric conditional variance models (depending on positive or negative value of residual) better fit the observed data than models with symmetric conditional variance.
- Negative value of coefficient estimate of the variable denoting asymmetric response means the presence of leverage effect. ‘Good’ news (negative residuals in the models) reduces the spot rate volatility with the leverage effect being proportional to squared residual value. The latter is supported by statistically significant estimate of corresponding in the threshold model, whereas the coefficient estimate in the exponential model is statistically insignificant at 5% level.

[20] Fornari and Mele investigated the first-order autoregressive process of spot rate in the markets in the USA and the UK. The coefficient estimates were close to unity, i.e. they found an evidence of temporary deviations from long-term mean value, while we may note a tendency to mean-reverting only.

[21] Fornari and Mele got the sums of coefficients estimates being less than unity. Therefore, in the developed financial markets spot rate volatility remains constant during long periods of time.

## §4. Parametric nonlinear model estimated using GMM

The estimates given above are estimates of parameters of discrete version of stochastic process modelled as an autoregressive time series model with autoregressive conditional variance of residuals. However, in theoretical models of term structure of interest rates the spot rate dynamics is specified as an continuous stochastic process:

$$dr = (a + br)dt + sr^{g}\,dz\,, \tag{6}$$

where $dz$ – the Wiener process increments, $a$, $b$, $s$, $g$ – constants[22].

Another discrete form of stochastic process (6) proposed by Chan et *al.*, has the general form written as:

$$\begin{aligned} & r_{t+1} - r_t = a + br_t + e_{t+1} \\ & E[e_{t+1}] = 0 \\ & E[e_{t+1}^2] = s^2 r_t^{2g} \end{aligned} \quad . \tag{7}$$

The parameters $a$, $b$, $s$, $g$ are estimated using the GMM under given constraints. In Table 3 we present restrictions on values of stochastic process parameters (6, 7) imposed by alternative models of term structure of interest rates.

In our case the GMM orthogonality condition can be written as $E[\mathbf{z}' f(\theta)] = 0$, where $\theta = \{a \quad b \quad s^2 \quad g\}$ – a set of estimated parameters; $f(\theta) = \begin{bmatrix} e_{t+1} \\ e_{t+1}^2 - s^2 r_t^{2g} \end{bmatrix}$ – a vector of the first two moments subject to $e_{t+1} = r_{t+1} - r_t - a - br_t$; $\mathbf{z}' = [1 \quad r_t]$ – a vector of instrumental variables which are orthogonal to the moments. The GMM estimates of models 1–9 are given in Table 4.

The results enable to draw a number of conclusions concerning the pattern of the GKO spot rate stochastic process.

Comparative analysis of alternative stochastic models of term structure of interest rates shows (based on J-statistics) that there is only one specification of spot rate stochastic process which cannot be rejected for the Russian market at 5% significance level. This is the square-root process proposed in the Cox-Ingersoll-Ross 1985 model. Other models are rejected as misspecified or overidentificated (the number of imposed constraints is higher than the number of valid ones)[23].

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[22] Kan and Zhang (Kan, R., Chu Zhang (1999) 'GMM tests of stochastic discount factor models with useless factors', *Journal of Financial Economics*, 54, pp. 103–127) showed that an inclusion in stochastic process specification additional exogenous factors (e.g., inflation rate or other macroeconomic variables) can lead to a bias toward zero in GMM estimates.

[23] Estimates of the model without constraints are given for reference and the model cannot be consider as a special specification of the process, since it has no analytical solution for bond prices.

Our results are different from conclusions given in papers by Chan et *al*., Pearson - Sun and Boero-Torricelli[24]. In those models the Cox-Ingersoll-Ross 1985 model was rejected for the US Treasury bills market in comparison with alternative specifications of stochastic process, namely, in favour of the Dothan 1978 model and the Cox-Ingersoll-Ross 1980 model. However, the results similar to our ones were got in studies by Brown, Schaefer, Johnson, de Munnik, Schotman, Dahlquist[25] who analysed the European bond markets (the UK, Germany, Denmark, Sweden, the Netherlands).

The mean-reverting models (Vasicek, Brennan-Schwartz, Cox-Ingersoll-Ross 1985) have better quality of describing the pattern of the GKO spot rate process compared to alternative specifications (measured by the share of explained variance of increments and squares of increments (volatility) of the spot rate). The result coincides with the conclusions withdrawn earlier from estimates of nonlinear time series models with autoregressive conditional variance of residuals.

The long-term mean value of the GKO spot rate calculated based on $a$ and $b$ estimates in the Cox-Ingersoll-Ross 1985 model is 40.5% annualised. That is slightly higher than the simple average of the time series (39.75%). The value of speed of mean-reverting is rather low (0.227), though it is statistically different from zero. The doubled product of mean value of spot rate and speed of mean-reverting is higher than the square of estimated standard deviation of the process ($s^2$). According to theoretical conclusions from the Cox-Ingersoll-Ross 1985 model, the result means that the upward drift in the stochastic process is too strong, and spot rate cannot return to a starting value. The inference supports our assumption on presence of only a tendency to mean-reverting in the spot rate pattern, but, the spot rate can hardly revert to the mean value in case of intensive shocks.

[24] Chan, K., G. A. Karolyi, F. Longstaff, A. Sanders (1992) 'An empirical comparison of alternative models of the short-term interest rate', *Journal of Finance*, 47, pp. 1209 – 1227; Pearson, N., Tong-Sheng Sun (1994) 'Exploiting the conditional density in estimating the term structure: An application to the Cox, Ingersoll, and Ross model', *Journal of Finance*, 49, pp. 1279 – 1304; Boero, G., C. Torricelli (1996) 'A comparative evaluation of alternative models of the term structure of interest rates', *European Journal of Operational Research*, 93, pp. 205 – 223.

[25] Brown, R., S. Schaefer (1994) 'The term structure of real interest rates and the Cox, Ingersoll, and Ross model', *Journal of Financial Economics*, 35, pp. 3 – 42; Brown, R., S. Schaefer (1996) 'Ten years of the real term structure: 1984 – 1994', *Journal of Fixed Income*, 5, pp. 6 – 22; Johnson, D. (1993) 'International interest rate linkages in the term structure', *Journal of Money, Credit, and Banking*, 25, pp. 755 – 770; De Munnik, J., P. Schotman (1994) 'Cross-section versus time series estimation of term structure models: Empirical results for the Dutch bond market', *Journal of Banking and Finance*, 18, pp. 997 – 1025; Dahlquist, M. (1995) 'Essays on the term structure of interest rates and monetary policy', *PhD. thesis (*Institute for International Economic Studies, University of Stockholm). Here we should once again note that we do not consider arbitrage-free stochastic models, which are more accurate in approximation of observed spot rates in comparison with the best single-factor and general equilibrium models of term structure (see., e.g., Raj, M., A. B. Sim, D. Thurston (1997) 'A generalized method of moments comparison of the Cox-Ingersoll-Ross and Heath-Jarrow-Morton models', *Journal of Economics and Business*, 49, pp. 169 – 192).

The variance of the spot rate increments is proportional to square root of the spot rate level[26]. Therefore, as the level increases, the variance of the stochastic process rises too, and the process is non-stationary. However, the variance grows slower than the spot rate level. That explains the lack of convergence in temporary (short-term) fluctuations of the variance in the component model.

Adequacy of the Cox-Ingersoll-Ross 1985 model to the Russian GKO/OFZ market is also supported by realisation of main theoretical inferences from the model concerning behaviour of the GKO yield curve. A rise in current spot rate induces an increase in yields to maturity on bills of all maturities, however, the impact on short end is stronger. An increase in long-term mean value of spot rate causes an increase in all yields, but the effect on long end of yield curve is stronger. These conclusions are supported by estimates of goodness-of-fit of analytical yield curves: in 70 of 205 observations the root-mean-square percent errors[27] are within 25%, and only in 42 cases the deviation makes up more than 50% of values of actual yield curve.

## §5. Stochastic volatility models

Stochastic volatility models imply the spot rate following a two-dimensional vector stochastic process, which has stochastic terms both in spot rate increments and its volatility. We consider two specification:

1) Kalman filter model

$$\begin{aligned} r_t &= (c + a_1 r_{t-1} + a_2 r_{t-2}) + s_{t-1} r_{t-1}^{g} e_{1,t} \\ \ln s_t^2 - m &= b(\ln s_{t-1}^2 - m) + x e_{2,t} \end{aligned} \qquad (8)$$

2) QML model

$$\begin{aligned} r_t &= c + a_1 r_{t-1} + a_2 r_{t-2} + e_t \\ e_t &\sim (0, s^2 r_t^{2g}) \end{aligned} . \qquad (9)$$

The difference between the two approaches is that the first one assumes mean-reverting volatility, while the second one refers to an extended version of GARCH model.

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[26] The estimates of models without constraints on parameter *g* (the model without any constraints and the Cox-Ross model) show that the estimated value of the parameter is close to 0.5.

[27] $RMSPE_t = \sqrt{\frac{1}{T_t} \sum_{n=1}^{12} \left( \frac{r_t^n - i_t^n}{i_t^n} \right)}$, where $3 \le T_t \le 12$ – the number of observations along yield curve (monthly terms) at time *t*, $r_t^n$ and $i_t^n$ – analytical and actual, correspondingly, values of bill yields with maturity *n*.

Estimation procedure for the model (8) has two steps. At the first step we estimate a simple AR(2) model to obtain the initial values of coefficients $c, a_1, a_2$ and residuals. At the second step we reformulate the model in the following way:

$$\begin{aligned} \ln(e_t^2) &= \ln s_{t-1}^2 + 2g \ln r_{t-1} + \ln(e_{1,t}^2) \\ \ln s_t^2 - m &= b(\ln s_{t-1}^2 - m) + xe_{2,t} \end{aligned},$$

where we assume $e_1, e_2$ being uncorrelated.

The likelihood function for the model (9) is

$$L(c, a_1, a_2, s^2, g) = \sum_i l_i(c, a_1, a_2, s^2, g),$$

where the individual contributions are given by

$$l_i(c, a_1, a_2, s^2, g) = \log f\left(\frac{r_t - c - a_1 r_{t-1} - a_2 r_{t-2}}{\sqrt{s^2 r_t^{2g}}}\right) - \tfrac{1}{2}\log(s^2 r_t^{2g}).$$

Estimates of models (8) and (9) are given in Table 5. The stochastic volatility models give some evidence on the presence of stochastic properties in the spot rate volatility. However, the hypothesis on mean-reverting in volatility tends to be rejected by the Kalman filter model. Coefficients $b$ and $x$ are insignificantly different from zero, the variance of error in volatility equation $s^2(e_2)$ also is zero. That supports the conclusions withdrawn from Component GARCH model (5).

The spot rate parameters estimates in model (9) are close to the ones obtained by GMM method. Estimates of parameter g in the model is about 0.72; that is higher than in unrestricted GMM model and the Cox-Ingersoll-Ross model. However, it is still less than one and it means the volatility rises with spot rate, but less than proportional one by one.

We consider the QML parameters estimates as the true parameters of the process. First, the specification of the process is more precise (the second autoregressive term is included). Second, we attribute the differences in estimates at least partly to statistical properties of the both methods. As our sample is not very large, the QML estimates are more efficient.

## Conclusions

We have analysed dynamics of the GKO spot rate process. Our results infer a number of conclusions concerning the pattern of the GKO market development and behaviour of term structure of interest rates in the transition economy. The most general inference is that pattern of spot rate process in transitional economy can be nested in existing theoretical model of term structure of interest rates.

The Russian market for government securities by its features is closer to the European financial markets compared to the market for US Treasury bills. This conclusion is supported by estimates of parameters of the GKO spot rate stochastic process using both the GMM and QML estimates of spot rate nonlinear models. The Cox-Ingersoll-Ross 1985 model of term structure of interest rates is the most adequate for the Russian GKO market. Many studies showed that it works well in the European financial studies, but is likely to be rejected for the US Treasury bill market.

The behaviour of the GKO yield term structure in 1994 through 1998 did not contradict to theoretical conclusions from the model; analytical yield curves have satisfactory accuracy of approximation of actual GKO yield curves.

The spot rate stochastic process is mean-reverting, but its variance although being stochastic does not exhibit mean-reverting property (according to Kalman filter estimates). The stochastic nature of spot rate volatility origins from different responses to ‘good’ and ‘bad’ news and a proportion to current spot rate level (but less than one by one).

## *Figures and tables*

***Table 1.***

| | Time period, number of observations | Dickey-Fuller test | Philips-Perron test | Enders-Granger test | Asymmetry of the process |
|---|---|---|---|---|---|
| $r_t$ | 12.9.94–16.8.98, 205 | -4.60 | -4.74 | 18.52 | 34.24 |

***Figure 1.*** ACF and PACF of the GKO spot rate**.**

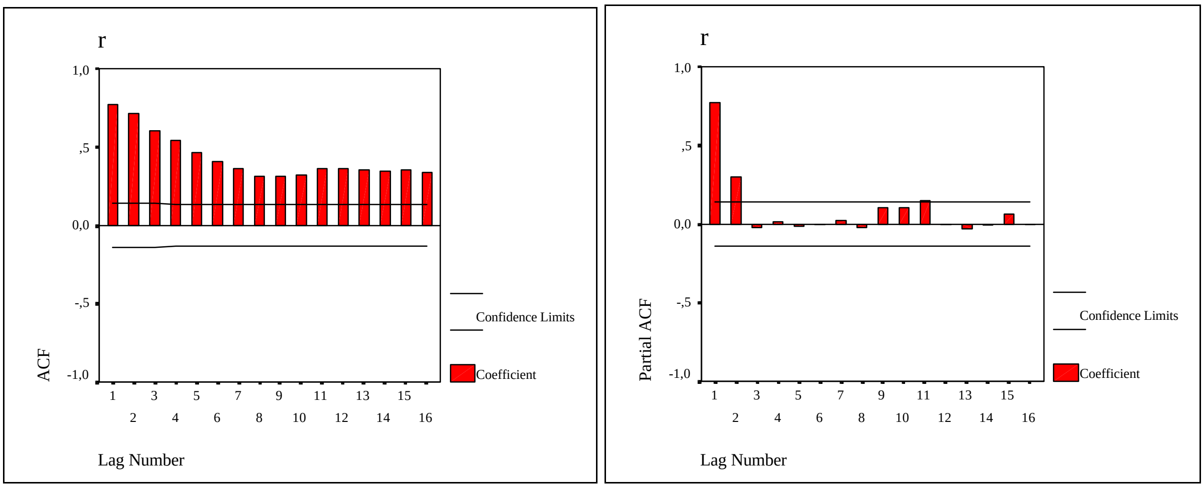


***Table 2.***

| Equation | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| ***c*** | 0.381<br>(4.54) | 0.222<br>(5.71) | 0.264<br>(4.70) | 0.275<br>(5.79) | 0.236<br>(5.05) |
| $a_1$ | 0.516<br>(7.84) | 0.527<br>(5.77) | 0.609<br>(8.40) | 0.522<br>(6.03) | 0.604<br>(8.06) |
| $a_2$ | 0.319<br>(4.86) | 0.301<br>(4.77) | 0.218<br>(3.18) | 0.262<br>(3.65) | 0.216<br>(3.68) |
| ***d*** | – | 0.001<br>(2.18) | 0.001<br>(3.44) | -0.401<br>(-3.20) | 0.100<br>(1.70) |
| ***a*** | – | 0.530<br>(4.89) | 0.397<br>(3.80) | -0.037<br>(-0.59) | 0.247<br>(6.62) |
| ***b*** | – | 0.635<br>(11.12) | 0.813<br>(20.63) | 0.900<br>(25.90) | 0.741<br>(18.54) |
| ***g*** | – | – | -0.609<br>(-4.48) | 0.420<br>(6.47) | -0.695<br>(-32.58) |
| ***f*** | – | – | – | – | 0.120<br>(8.38) |
| **AIC** | -0.395 | -0.772 | -0.874 | -0.929 | -0.773 |
| **BIC** | -0.346 | -0.674 | -0.760 | -0.815 | -0.642 |

* The table shows estimates and *t*-statistics (in parentheses) of corresponding coefficients and values of information criteria for comparison of the model performance.

***Table 3.***

| 1 | Model | *a* | *b* | $s^2$ | *g* |
|---|---|---|---|---|---|
| 1 | Without constraints | – | – | – | – |
| 2 | Merton (Merton, 1973)[28] | – | 0 | – | 0 |
| 3 | Cox-Ross (Cox, Ross, 1976)[29] | 0 | – | – | – |
| 4 | Vasicek (Vasicek, 1977)[30] | – | – | – | 0 |
| 5 | Dothan (Dothan, 1978)[31] | 0 | 0 | – | 1 |
| 6 | Cox-Ingersoll-Ross (Cox, Ingersoll, Ross, 1980)[32] | 0 | 0 | – | $^3/_2$ |
| 7 | Brennan-Schwartz (Brennan-Schwartz, 1982)[33] | – | – | – | 1 |
| 8 | Marsh-Rosenfeld (Marsh, Rosenfeld, 1983)[34] | 0 | – | – | 1 |
| 9 | Cox-Ingersoll-Ross (Cox, Ingersoll, Ross, 1985)[35] | – | – | – | $^1/_2$ |

***Table 4.***

| Model | *a* | *b* | $s^2$ | *g* | J-statistics | $R_1^2$ | $R_2^2$ |
|---|---|---|---|---|---|---|---|
| 1 | 0.092<br>(3.35) | -0.227<br>(-3.09) | 0.118<br>(5.64) | 0.567<br>(5.92) | 0.000* | 0.144 | 0.440 |
| 2 | -0.009<br>(-1.11) | 0 | 0.050<br>(6.51) | 0 | 30.600 | -0.002 | -0.062 |
| 3 | 0 | -0.092<br>(-3.44) | 0.125<br>(5.32) | 0.555<br>(4.95) | 28.560 | 0.050 | 0.214 |
| 4 | 0.094<br>(3.41) | -0.231<br>(-3.17) | 0.044<br>(5.98) | 0 | 23.664 | 0.107 | 0.225 |
| 5 | 0 | 0 | 0.134<br>(4.45) | 1 | 13.056 | 0.047 | 0.149 |
| 6 | 0 | 0 | 0.101<br>(2.60) | $^3/_2$ | 14.688 | -0.001 | -0.093 |
| 7 | 0.091<br>(3.38) | -0.226<br>(-3.08) | 0.107<br>(4.15) | 1 | 13.260 | 0.110 | 0.275 |
| 8 | 0 | -0.089<br>(-3.52) | 0.114<br>(5.41) | 1 | 21.624 | 0.024 | 0.183 |
| 9 | 0.092<br>(3.36) | -0.227<br>(-3.15) | 0.115<br>(7.10) | $^1/_2$ | 1.020* | 0.114 | 0.427 |

* The hypothesis on validity of constraints is not rejected at 5% significance level. $R_1^2$ shows the explained share of variance of spot rate increments and $R_2^2$ shows the explained share of variance of squared increments (volatility) of spot rate.

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***Table 5*.***

| | **Model (8)** | **Model (9)** |
|---|---|---|
| $c$ | 0.063<br>(0.084) | 0.042<br>(0.023) |
| $a_1$ | 0.516<br>(0.066) | 0.618<br>(7.34) |
| $a_2$ | 0.319<br>(0.066) | 0.277<br>(0.054) |
| $s^2$ | 0.032 | 0.128<br>(0.024) |
| $g$ | 0.726<br>(0.132) | 0.718<br>(0.054) |
| $m$ | -3.429<br>(0.370) | – |
| $b$ | -0.031<br>(221.6) | – |
| $x$ | 0.179<br>(0.476) | – |
| $s^2(e_1)$ | 3.65<br>(10.26) | – |
| $s^2(e_2)$ | 0.233<br>($11\cdot10^7$) | – |
| **LogLikelihood** | -438.7 | 91.8 |
| **AIC** | – | -0.855 |
| **BIC** | – | -0.774 |

* The table shows estimates and standard errors of corresponding coefficients.